Contents

Preface to the Second Edition v

Preface to the First Edition vii

Acknowledgments xi

1 Physical Quantities 1

1.1 Introduction 2
1.2 Quantities and Units 3
1.3 Powers of 10 5
1.4 Accuracy of Numbers 6
Problems 7

2 Vectors 9

2.1 Introduction 10
2.2 Vector Components 10
2.3 Unit Vectors 14
2.4 Dot Product 15
2.5 Cross Product 16
Problems 18

3 Uniformly Accelerated Motion 21

3.1 Introduction 22
3.2 Speed and Velocity 22
3.3 Acceleration 24
3.4 Linear Motion 25
3.5 Projectile Motion 30
Problems 34

4 Newton's Laws 37

4.1 Introduction 38
4.2 Newton's Laws 38
4.3 Mass 41
4.4 Weight 42
4.5 Applications of Newton's Laws 43
4.6 Friction 48
Problems 50

5 **Work, Energy, and Power** 55

5.1 Introduction 56
5.2 Work 56
5.3 Potential Energy 58
5.4 Work Done by a Variable Force 59
5.5 Kinetic Energy 60
5.6 Energy Conservation 61
5.7 Power 64
Problems 65

6 **Momentum and Collisions** 69

6.1 Introduction 70
6.2 Center of Mass 70
6.3 Motion of the Center of Mass 73
6.4 Momentum and its Conservation 74
6.5 Collisions 77
Problems 80

7 **Rotational Motion** 83

7.1 Introduction 84
7.2 Measurement of Rotation 84
7.3 Rotational Motion 85
7.4 Equations of Rotational Motion 88
7.5 Radial Acceleration 90
7.6 Centripetal Force 91
7.7 Orbital Motion and Gravitation 93
Problems 95

8 **Rotational Dynamics** 99

8.1 Introduction 100
8.2 Moment of Inertia and Torque 100
8.3 Rotational Kinetic Energy 103
8.4 Power 107
8.5 Angular Momentum 108
8.6 Conservation of Angular Momentum 110
Problems 111

9 **Kinetic Theory of Gases and the Concept of Temperature** 115

9.1 Introduction 116
9.2 Molecular Weight 116

9.3 Thermometers 117
9.4 Ideal Gas Law and Absolute Temperature 118
9.5 Kinetic Theory of Gas Pressure 121
9.6 Kinetic Theory of Temperature 123
9.7 Measurement of Heat 125
9.8 Specific Heats of Gases 126
9.9 Work Done by a Gas 127
9.10 First Law of Thermodynamics 128
Supplement 9-1: Maxwell-Boltzmann Statistical Distribution 129
Problems 132

10 Oscillatory Motion 135

10.1 Introduction 136
10.2 Characterization of Springs 136
10.3 Frequency and Period 137
10.4 Amplitude and Phase Angle 138
10.5 Oscillation of a Spring 138
10.6 Energy of Oscillation 144
Problems 146

11 Wave Motion 149

11.1 Introduction 150
11.2 Wavelength, Velocity, Frequency, and Amplitude 150
11.3 Traveling Waves in a String 151
11.4 Energy Transfer of a Wave 155
Problems 158

12 Interference of Waves 161

12.1 Introduction 162
12.2 The Superposition Principle 162
12.3 Interference from Two Sources 163
12.4 Double Slit Interference of Light 166
12.5 Single Slit Diffraction 170
12.6 Resolving Power 172
12.7 X-Ray Diffraction by Crystals: Bragg Scattering 174
12.8 Standing Waves 177
Problems 178

13 Electrostatics 181

13.1 Introduction 182
13.2 Attraction and Repulsion of Charges 182

13.3 Coulomb's Law 183
13.4 Charge of an Electron 187
13.5 Superposition Principle 187
Problems 189

14 The Electric Field and the Electric Potential 191

14.1 Introduction 192
14.2 The Electric Field 192
14.3 Electrical Potential Energy 195
14.4 Electric Potential 198
14.5 The Electron Volt 201
14.6 Electromotive Force 201
14.7 Capacitance 202
Problems 203

15 Electric Current 207

15.1 Introduction 208
15.2 Motion of Charges in an Electric Field 208
15.3 Electric Current 209
15.4 Resistance and Resistivity 212
15.5 Resistances in Series and Parallel 214
15.6 Kirchhoff's Rules 218
15.7 Ammeters and Voltmeters 223
15.8 Power Dissipation by Resistors 225
15.9 Charging a Capacitor—RC Circuits 226
Problems 229

16 Magnetic Fields and Electromagnetic Waves 233

16.1 Introduction 234
16.2 Magnetic Fields 234
16.3 Force on Current-Carrying Wires 235
16.4 Torque on a Current Loop 236
16.5 Magnetic Dipole Moment 238
16.6 Force on a Moving Charge 240
16.7 The Hall Effect 241
16.8 Electromagnetic Waves: The Nature of Light 243
Problems 245

17 The Beginning of the Quantum Story 249

17.1 Introduction 250

1.1 INTRODUCTION

People have always observed natural phenomena and then verbalized their observations for discussion with others. With the development of *physics*, the words in the verbal descriptions were given symbols, which could then be manipulated according to the rules of mathematics.

In physics two fundamental processes are involved. The first is the description of natural phenomena based on experiments, which control variables. Theories are not accepted by physicists until verified by experiment. The second is mathematical manipulation or theorizing, which is a predictive process. If the observed quantities have been described properly and given the proper symbols, then the subsequent mathematical manipulations will result in new relationships that must be correct on testing, else their formulation will be rejected as incorrect or inadequate. Furthermore, the results of the relationships must stand the test of time. In this sense, time means enough time for many experiments to be performed to test the relationships. The laws presented in this book have met these requirements.

Although human beings have observed Nature from their first existence, it was not until the time of Galileo (1564–1642) that these observations began to be expressed in modern mathematical terms. Subsequent studies, measurements, and critical evaluations developed what we now call the *First Principles* of physics, which have truly stood the test of time. The first six chapters of this book discuss these principles and show how they are used in their simplest form. *Other areas of physics must satisfy these First Principles*. Chapters 7 through 16 illustrate the use of these principles in rotational motion, the behavior of gases, and electric and magnetic phenomena. Chapters 1 through 16 constitute what is usually known as *classical physics*. In the remaining chapters, we introduce a different way of describing the behavior of small physical particles. For example, although we may continue to consider the electron as the smallest negatively charged particle, experiments have shown that its behavior can also be described as a wave instead of a particle. The mathematics of waves, instead of that of particles, must be used to explain the electron's behavior in certain situations, whereas the mathematics of particles still applies in other situations. This revolution in thought, begun in the early part of this century, has led to the method, or science, of *wave mechanics*, which is more generally called *quantum mechanics*.

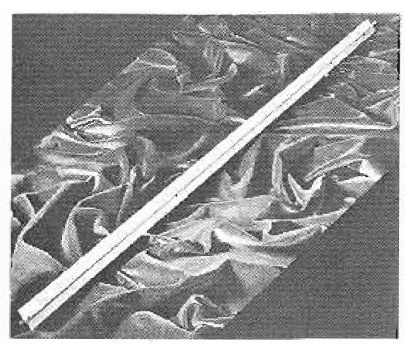
When the behavior of an electron within a solid is sought, very little can be learned by the particle treatment, but a vast amount of understanding can be achieved by the wave approach. How does this fit in with our mention of the test of time and observation? Although the actual length of time of this modern model (about 90 years) is short compared with the time since Galileo, the number of experiments that have been performed is far greater. It has been said that of all the physicists who have ever lived, 95% are still alive.

The observation requirement is somewhat more subtle. We cannot observe fundamental particles such as an electron in the same way that we observe macroscopic objects; they are too small. In fact, we will later discuss the *Uncertainty Principle*, which will show that the mere act of observing will change the state of the particle. Because of this principle, experiments that would completely characterize a small particle are too difficult to perform. What we do observe is the statistical behavior of a vast number of particles, and we infer the average behavior of a member of the statistical ensemble. Therefore, although we may never know the physical parameters of the individual particle, there are many physical experiments that can tell us if the statistical model is satisfactory.

1.2 QUANTITIES AND UNITS

If a physical phenomenon is to be quantified, there must be suitable, agreed-on measuring devices. Many measuring systems have been created in the past, none perfect. It is desirable to have a measurement system that has the least number of fundamental parameters. In classical physics, these are length, mass, and time. All mechanical quantities can be defined in terms of these three. For example, speed is the ratio of a length to a time. We then choose a standard for each of the fundamental parameters. Most scientific measurements use the metric system. There are two versions of the metric system in use, the *cgs* (centimeter, gram, second) and the *mks* (meter, kilogram, second). Although the cgs system is still often used in the biological sciences, most measurements by physicists now use the mks system. This is the mechanical part of the more general SI (Système Internationale) that covers all physical measurements. The English system of units (foot, pound, second) is often dictated by manufacturing specifications. We will mostly use SI units in this book.

The unit of a quantity is as important as the magnitude, as indicated in the quote from *The New York Times* at the beginning of this chapter. It is meaningless to say "the distance between two points is 10," because the 10 may be meters, miles, or inches. The units are an integral part of the measurement and must be treated algebraically. One may substitute for them or convert them to a different system, but they cannot be gotten rid of except by an algebraic process. For example, π is dimensionless, but it is defined on the basis of two measured quantities, the circumference of a circle divided by its diameter. It is independent of units because they cancel: circumference (meters)/diameter (meters), and it is seen that meters cancel algebraically.


United States copy of the original platinum-iridium bar which for many years was the standard of length: the meter.

Conversion of units must be done with care, and, in order to convert, a relationship between units of two different systems must be known. Let us illustrate this with a trivial example. How many feet are there in 5 mi?

$$5 \text{ mi} = ? \text{ ft} \qquad (1.1)$$

We know the relation

$$5280 \text{ ft} = 1 \text{ mi} \tag{1.2}$$

How do we substitute this with care? The safest rule is always to multiply by one (unity), because we know that in algebra, multiplication of anything by unity leaves it unchanged. Our conversion relation, Eq. 1.2, can be made into two different forms of unity depending on whether we divide both sides by miles or by feet:

$$\frac{5280 \text{ ft}}{1 \text{ mi}} = 1 \quad \frac{1 \text{ mi}}{5280 \text{ ft}} = 1$$

We can now multiply the left side of Eq. 1.1 by the first form of unity, which gives

$$5 \text{ mi} = 5 \cancel{\text{mi}} \times \frac{5280 \text{ ft}}{1 \cancel{\text{mi}}} = 5 \times 5280 \text{ ft} = 26,400 \text{ ft}$$

We see that miles in both the numerator and the denominator have cancelled algebraically, leaving feet as the unit.

We can do more than one algebraic step at a time. For example, how many seconds are in 1 day?

$$1 \text{ day} = ? \text{ sec}$$

We know three conversion relations

$$1 \text{ day} = 24 \text{ h} \quad 1 \text{ h} = 60 \text{ min} \quad 1 \text{ min} = 60 \text{ sec}$$

We select our choices of unity to give successive algebraic cancellations

$$1 \text{ day} = 1 \cancel{\text{day}} \left(\frac{24 \cancel{\text{h}}}{1 \cancel{\text{day}}}\right)\left(\frac{60 \cancel{\text{min}}}{1 \cancel{\text{h}}}\right)\left(\frac{60 \text{ sec}}{1 \cancel{\text{min}}}\right)$$

$$= 24 \times 60 \times 60 \text{ sec} = 84,600 \text{ sec}$$

We may convert two units simultaneously in a single equation to save a lot of writing. For example, a car traveling at 60 mi/h travels how many feet per second?

$$60 \frac{\text{mi}}{\text{h}} = 60 \frac{\cancel{\text{mi}}}{\cancel{\text{h}}} \left(\frac{5280 \text{ ft}}{1 \cancel{\text{mi}}}\right)\left(\frac{1 \cancel{\text{h}}}{60 \cancel{\text{min}}}\right)\left(\frac{1 \cancel{\text{min}}}{60 \text{ sec}}\right) = \frac{60 \times 5280}{60 \times 60} \text{ ft/sec} = 88 \text{ ft/sec}$$

Remember that in square or cubic units, all measurements must be in the same units. It makes no sense to calculate the area of a room if its length is measured in feet and its width is measured in meters. Both measurements should be in the same system. Also remember that when converting square or cubic units they must be squared or cubed just as algebraic quantities. For example, how many cubic centimeters (1 m = 100 cm) are there in a volume of 1 m^3?

$$1 \text{ m}^3 = 1 \cancel{\text{m}}^3 \left(\frac{100 \text{ cm}}{1 \cancel{\text{m}}}\right)\left(\frac{100 \text{ cm}}{1 \cancel{\text{m}}}\right)\left(\frac{100 \text{ cm}}{1 \cancel{\text{m}}}\right) = 1,000,000 \text{ cm}^3$$

1.3 POWERS OF 10

Often, very large and very small numbers arise in physics. In 1 cm^3 of a solid there are a vast number of atoms, about 1 followed by 21 zeros. Measurements have shown that the range of atomic diameters in meters is between 0.0000000001 and 0.0000000003 m. Because of the difficulty of reading and writing numbers with many zeros, we use powers of 10 notation. Recall the algebraic postulate that any quantity to the zeroth power is, identically, unity. A brief table of some powers of 10 follows.

$$10^0 = 1 \qquad 10^{-1} = \tfrac{1}{10} = 0.1$$

$$10^1 = 10 \qquad 10^{-2} = \tfrac{1}{100} = 0.01$$

$$10^2 = 100 \qquad 10^{-3} = \tfrac{1}{1000} = 0.001$$

$$10^3 = 1000 \qquad 10^{-4} = \tfrac{1}{10000} = 0.0001$$

$$10^4 = 10{,}000$$

Some of these are used as prefixes; for example, 10^3 = kilo, 10^{-3} = milli, 10^{-6} = micro, 10^{-9} = nano, 10^{-12} = pico.

Let us review the algebra of adding and multiplying powers with the letter a representing 10. (These rules apply for a equal to any value other than zero.)

EXAMPLE 1-1

$$(b \times a^n)(c \times a^m) = bca^n a^m = bca^{n+m}$$

Substitute arbitrary numbers for the letters; for example, $b = 2$, $c = 3$, $n = 4$, and $m = 2$.

$$(2 \times 10^4)(3 \times 10^2) = 2 \times 3 \times 10^4 \times 10^2 = 6 \times 10^6$$

EXAMPLE 1-2

$$\frac{e \times a^n}{f \times a^m} = \frac{e}{f} a^{n-m}$$

If $e = 6, f = 2, n = 4$, and $m = 2$,

$$\frac{6 \times 10^4}{2 \times 10^2} = \frac{6}{2} 10^{4-2} = 3 \times 10^2$$

EXAMPLE 1-3

$$(b \times a^n) + (c \times a^m)$$

This form can be simplified if $n = m$, then

$$(b \times a^n) + (c \times a^n) = (b + c)a^n$$

If $n \neq m$, they can be made equal. For example, let

$$b = 2 \text{ and } c = 3, \quad n = 4 \text{ and } m = 5$$

substituting in the algebraic relation, we have

$$2 \times 10^4 + 3 \times 10^5$$

But $10^5 = 10^1 \times 10^4$ and therefore

$$2 \times 10^4 + 3 \times 10^5 = 2 \times 10^4 + 3 \times 10^1 \times 10^4$$

$$= (2 + 30)10^4 = 32 \times 10^4 = 3.2 \times 10^5$$

This sum could also be done by converting the first term

$$2 \times \frac{10^1}{10^1} \times 10^4 + 3 \times 10^5 = 0.2 \times 10^5 + 3 \times 10^5 = 3.2 \times 10^5$$

In scientific notation, usually only one digit is placed in front of the decimal point.

1.4 ACCURACY OF NUMBERS

Suppose we wish to find the area of a rectangular surface. We know that we multiply the length l by the width w. Suppose we take a metric ruler to measure l and w. The metric ruler's smallest division is the millimeter, 10^{-3} m. Figure 1-1 illustrates the use of a metric ruler to measure the length of the rectangle. We can see that the measure of length lies between 47.6 and 47.7 cm; from the position of the edge of the rectangle we can estimate the second decimal as being less than 0.5 mm but not less than 0.3 mm. We can therefore express our measurement as 47.64 ± 0.01 cm, or 0.4764 ± 0.0001 m. Thus, the last digit is always uncertain and the $\pm$ value is the magnitude of the uncertainty. Suppose we have measured the width as 0.6343 ± 0.0001 m; what is the accuracy of the calculation of the area? We examine the two extremes. The largest area is

$$0.4765 \text{ m} \times 0.6344 \text{ m} = 0.3023 \text{ m}^2$$

and the smallest is

$$0.4763 \text{ m} \times 0.6342 \text{ m} = 0.3021 \text{ m}^2$$

We can write the answer as the average between the two values with $\pm$ the uncertainty, or 0.3022 ± 0.0001 m^2. Therefore, the accuracy of the product cannot exceed the accuracy of any of the components in the product.

This type of analysis must be extended to all data-handling, for example, sums, differences and quotients. Suppose that we want to sum two numbers known with

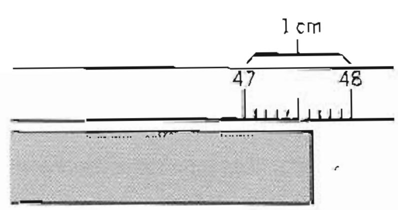


FIGURE 1-1

different degrees of accuracy. What is the accuracy of the sum? As an example, consider three successive points A, B, and C on a straight line. If the distance between A and B is 15.75 m and the distance between B and C is 2.432 m, what is the distance between A and C? The answer is obtained by summing the two numbers. If we use a calculator to evaluate the sum, it will tell us that the answer is 18.182 m. However, the only reliable answer is 18.18 m, because we do not know what the third digit after the decimal point of the first number (15.75 m) is. No matter how accurately a given parameter is measured, when it is combined arithmetically with another measurement the result is only as accurate as the least-accurate measurement. The number of accurate figures in a measurement is called the number of *significant figures*. Computer science students are often misled because computer or calculator answers may have 8 to 10 figures. In general, most of these figures have no significance and the answer should be rounded off to the lowest number of significant figures of the quantities used in the calculation. It is *incorrect* to give an answer with a greater number of figures.

PROBLEMS

1.1 Express your height in meters, using the relation 1 in. = 2.54 cm.

1.2 Use the relations 1 mi = 5280 ft and 1 m = 3.28 ft to express the speed 60 mi/h in meters per second (m/sec).

1.3 How many kilometers are in 1 mi?

1.4 Express the following in scientific notation (a single digit to the left of the decimal):
0.038, 0.000042, 5280, 62.356,
$(4 \times 10^3 + 3 \times 10^2)6 \times 10^{-3}$
$3.2 \times 10^4/(6.1 \times 10^{-2} + 9.2 \times 10^{-3})$

1.5 Express the following operation in scientific notation

$$\frac{4 \times 10^2 + 6 \times 10^3}{2 \times 10^{-4}}$$

1.6 Light travels at 186,000 mi/sec. Assume an average of 365 days in a year. (a) How many years does it take the light to reach us from the sun, which is 9.3×10^7 mi from the earth? (b) How many years does it take the light to reach us from the nearest star, other than the sun, which is 1.8×10^{13} mi from us?

1.7 Astronomers measure large distances in a unit called the *light-year*. This is the distance that light traveling at approximately 186,000 mi/sec will travel in 1 yr. How many miles are in 1 light-year?

1.8 Assume that the average lecture period is 1 micro-century (10^{-6} centuries); how long is the lecture period in minutes?

Answer: 52.6 min.

1.9 A *light-fermi* is a unit of time proposed by science-fiction writer Isaac Asimov. It is defined as the time taken by light to travel the distance of 1 fermi (10^{-15} m), which is the approximate size of the proton. How long is a light-fermi in seconds? Light travels at 3×10^8 m/sec.

1.10 There are approximately 8×10^{28} copper atoms in 1 m^3 of copper. (a) What is the volume occupied by a copper atom? (b) What is the radius of a sphere having that volume? (volume of a sphere = $4\pi r^3/3$)

1.11 Assume that atoms have spherical shape with average radius 4×10^{-10} m. How many atoms are there in the earth? Neglect the volume lost in packing the spheres and take the average radius of the earth to be 6.37×10^6 m.

Answer: 4.04×10^{48}.

1.12 In the Old Testament the Lord commanded Noah to build an ark 300 cubits long, 50 cubits wide, and 30 cubits high. A cubit is the length from a man's elbow to the tip of his extended middle finger. We do not know Noah's height, so measure a cubit from both a short person and a tall person. Assume the ark was a parallelepiped with right angles. (a) What are the maximum and the minimum values of its volume? (b) Assuming that the average animal required a space of $2 \times 4 \times 6$ ft^3, and that one half the volume of the ark was for food and passengers, what is the possible variation in the number of animals that could be accommodated?

1.13 Density is defined as the mass per unit volume. Take the average density of the earth to be 5.5 g/cm^3 and assume that the earth is a sphere of radius 6.37×10^3 km. Calculate the mass of the earth.

Answer: 5.96×10^{24} kg.

1.14 A neutron is one of the constituent particles of the nucleus. The mass of the neutron is 1.67×10^{-27} kg. Assuming that the neutron is a sphere of radius 1 F (10^{-15} m), what is the density of the neutron in g/cm^3? Compare your answer with the average density of the earth (see problem 1.13).

1.15 The radius of a carbon atom is about 2.5×10^{-8} cm. (a) How many could fit in a row 1-cm long? (b) How many could fit in a layer one atom deep and area 1 cm^2? (c) How many could fit in a cube 1 cm on each side? (d) If a crystal of carbon atoms (diamond) had this form, what is the minimum number of impurity atoms that could block the light coming through the faces of a 1-cm^3 cube? Express your answer in both percent and in parts per million. (*Hint*: Assume as an approximation that a layer of impurity atoms on each of three faces of the cube at right angles to each other could block all the light.)

Answer: (a) 2×10^7 cm^{-1}, (b) 4×10^{14} cm^{-2}, (c) 8×10^{21} cm^{-3}, (d) 1.2×10^{15}, 1.5×10^{-5}%, 0.15 ppm.

1.16 The distance x of an object from a certain origin is found to vary with time t as $x = a_1 + a_2 t + a_3 t^2$, where x is in meters, t is in seconds, and a_1, a_2, and a_3 are constants. What are the units of a_1, a_2, a_3?

1.17 A student is trying to find what parameters determine the period (time for a full swing) of a pendulum. After some experimentation, he concludes that the period T is given by $T = 2\pi g/l$ where $g = 9.8$ m/s^2 is the acceleration of free-falling bodies near the surface of the earth and l is the length of the pendulum. (a) Show by the units of the terms that the student's conclusion is incorrect. (b) Assuming that the period depends only on g and l, what is the proper functional dependence of T on these two quantities?

2.1 INTRODUCTION

We will be dealing with two types of quantities in this book. Some quantities are fully specified only by a number and a unit, such as a quart of milk or a pound of potatoes. Such a quantity consisting only of magnitude is called a *scalar* quantity. Other measurements have meaning only if direction is specified along with the magnitude. For example, telling a stranger that a gas station is 1 mi away will not help him unless you specify the direction also. A quantity that has both magnitude and direction and obeys certain algebraic laws is called a *vector* quantity and will be indicated in this book by boldface type.

2.2 VECTOR COMPONENTS

A vector direction must be specified in relation to a given coordinate system. Given a coordinate system, any vector can be expressed in terms of its components.

Let us examine the concept of vector components first by simply using the compass points north, south, east, and west as the directions.

Suppose you walk 5.0 mi east and then 4.0 mi north. How far are you and in what direction from the starting point?

We draw Fig. 2-1. We see that we have right-angle geometry and, because R is the hypotenuse, using the pythagorean theorem, we have

$$R = \sqrt{(4.0 \text{ mi})^2 + (5.0 \text{ mi})^2} = 6.4 \text{ mi}$$

We also know from trigonometry that

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4.0 \text{ mi}}{5.0 \text{ mi}} = 0.8$$

or

$$\theta = \arctan 0.8 = 39^\circ$$

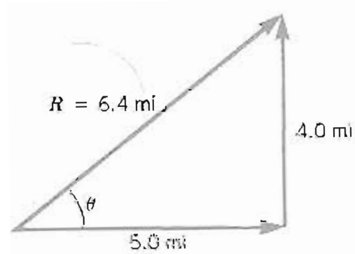


FIGURE 2-1

The distance R together with its orientation θ is called the *vector sum* of the two vectors 5.0 mi east and 4.0 mi north and is given the name *resultant*. The 5.0 mi east vector and the 4.0 mile north vector are called the *components* of R.

Suppose, instead, you walk 5.0 mi east and 4.0 mi northeast, namely, 45° north of east as in Fig. 2-2. What is the resultant?

This is a little more complicated and, although you could use the law of cosines and the law of sines to solve the problem, there is a simpler way that will be used throughout the book. This method is particularly useful when you deal with problems that involve more than two vectors. Let us reduce the problem to two questions.

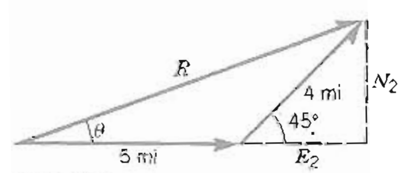


FIGURE 2-2

1 How far are you to the east of the starting point?
2 How far are you north of the starting point?

Note that the additional dashed lines in Fig. 2-2 labeled E_2 and N_2 represent the respective distances east and north traveled in the second leg. In this second part of the walk we see again a right triangle with E_2 and N_2 as the legs and with 4.0 mi as the hypotenuse. Recall from trigonometry that $\sin 45^\circ = N_2/4.0$ mi and $\cos 45^\circ = E_2/4.0$ mi. Therefore, $N_2 = (4.0 \text{ mi}) \sin 45^\circ$ and $E_2 = (4.0 \text{ mi}) \cos 45^\circ$. Now make a table of the data

		East		*North*
Walk 1		5 mi		0 mi
Walk 2	$(4.0 \text{ mi}) \cos 45^\circ =$	2.8 mi	$(4.0 \text{ mi}) \cos 45^\circ =$	2.8 mi
	Total	7.8 mi		2.8 mi

Take a fresh piece of graph paper and plot these distances from a starting point, which will be at the origin of a compass coordinate system, as in Fig. 2-3. The point marked x is your location from the starting point, R is the distance, and θ is the angle. We now have right-triangle geometry again and may write as before

$$R = \sqrt{(7.8 \text{ mi})^2 + (2.8 \text{ mi})^2} = 8.3 \text{ mi}$$

$$\theta = \arctan\left(\frac{2.8 \text{ mi}}{7.8 \text{ mi}}\right) = 19.7^\circ$$

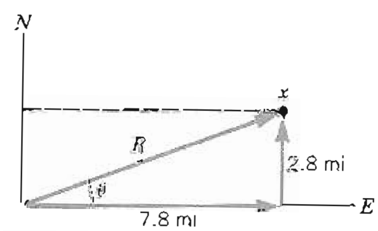


FIGURE 2-3

Consider the more complicated walk of Fig. 2-4. What is the resultant R of the four displacements shown?

To find R and θ, make a table of east-west and north-south displacements. Note here that the table will be in terms of east and north, so that a displacement to the west will be a negative east displacement and one to the south will be negative north. The components of the 4.0 mi, 5.0 mi, 7.0 mi, and 2.0 mi displacements can be found by putting pieces of graph paper with the origins at the starting points of these legs and finding the components by right-triangle geometry, as in Fig. 2-5.

Construct a table as in the previous example.

		East		*North*
Walk 1	$(4.0 \text{ mi}) \cos 30^\circ =$	3.5 mi	$(4.0 \text{ mi}) \sin 30^\circ =$	2.0 mi
Walk 2	$(5.0 \text{ mi}) \cos 60^\circ =$	2.5 mi	$(5.0 \text{ mi}) \sin 60^\circ =$	4.3 mi
Walk 3	$-7.0 \text{ mi} =$	-7.0 mi	$0 \text{ mi} =$	0 mi
Walk 4	$-(2.0 \text{ mi}) \cos 45^\circ =$	-1.4 mi	$-(2.0 \text{ mi}) \sin 45^\circ =$	-1.4 mi
	Total	-2.4 mi		4.9 mi

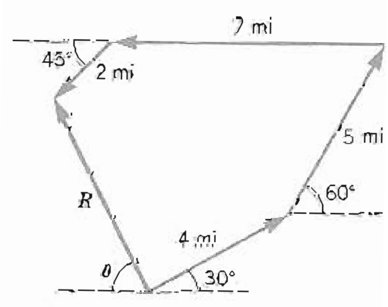


FIGURE 2-4

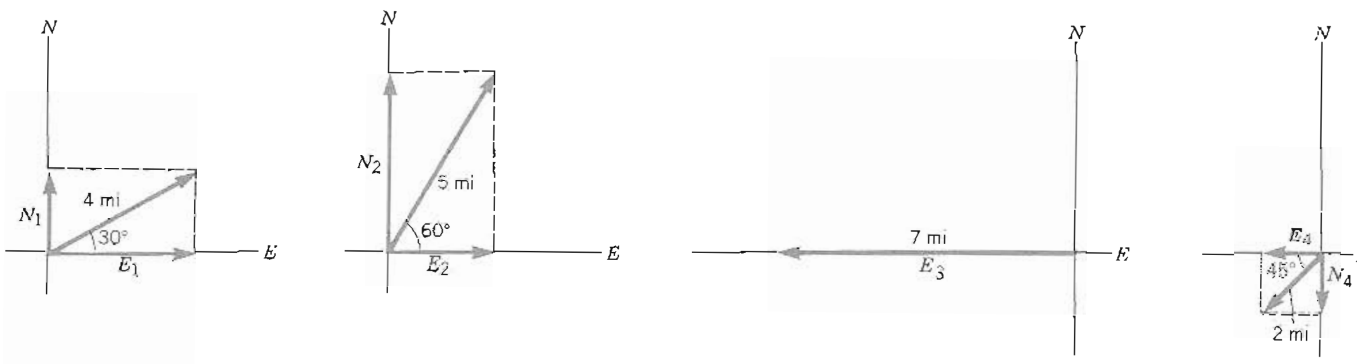


FIGURE 2-5

Now take a piece of graph paper and plot the total east and north displacements as in Fig. 2-6.

$$R = \sqrt{(-2.4\text{ mi})^2 + (4.9\text{ mi})^2} = 5.5\text{ mi}$$

$$\theta = \arctan\left(\frac{4.9\text{ mi}}{|-2.4\text{ mi}|}\right) = 63.9^\circ\text{ north of the west direction}$$

What we have done in Fig. 2-6 is to define an angle θ as less than 90°. This makes both the calculation and the spatial location much simpler. To work with $\theta < 90^\circ$ we had to ignore the sign of the coordinate and locate the resulting angle on the graph. Had we kept the sign of the coordinate in the calculation of the angle, it would not have helped much because the tangent is negative in two of the quadrants. We would still have to rely on some construction to locate the angle. The method shown in the preceding example, which first uses a graph to show where you are, leaves no question as to the meaning of the angle θ. How does one specify the angle? It is equally correct to say it is 64° north of west or, using the 360° scale with east as 0°, the angle would be $180^\circ - 64^\circ = 116^\circ$. One other point should be noted. In Fig. 2-5 we symbolically used four pieces of graph paper to obtain the components of the vectors. We could have equally used a single piece by putting the beginning of each of the walks at the origin of the graph paper as in Fig. 2-7; henceforth, we will conserve paper by this method.

Now that we have related coordinate systems to navigation, we can apply the same techniques to cartesian coordinates x-y instead of compass directions. To show how the vector component method is used in more complicated situations, we will abandon our walks and consider forces. The concept of forces will be developed more fully in Chapter 4. For now, we can simply rely on our experience that a force is that which when exerted in some direction against an object may or may not cause it to move. It is a vector quantity.

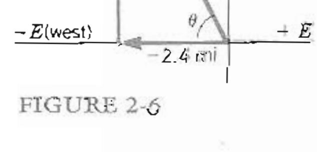


FIGURE 2-6

EXAMPLE 2-1 A box is pulled by two persons exerting the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ shown in Fig. 2-8, where $\mathbf{F}_1$ is given as 50 lb. Two questions may now be asked. 1.

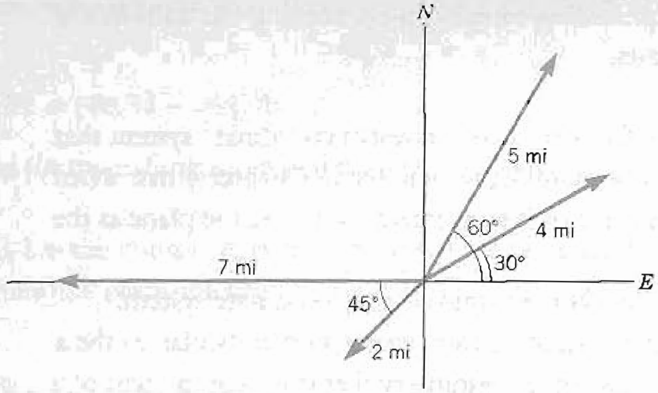


FIGURE 2-7

What force F_2 must be applied so that the box moves only in the x direction? 2. What single force could replace F_1 and F_2 so that the box moves only in the x direction?

Solution We obtain answers to these two questions by first constructing a vector diagram of the forces, as in Fig. 2-9, and tabulating the components of the forces.

Force	*x components*	*y components*
F_1	$(50\text{ lb})\cos 30° = 43.3\text{ lb}$	$-(50\text{ lb})\sin 30° = -25.0\text{ lb}$
F_2	$F_2 \cos 37° = 0.8F_2$	$F_2 \sin 37° = 0.6F_2$
Total	$43.3\text{ lb} + 0.8F_2$	$-25.0\text{ lb} + 0.6F_2$

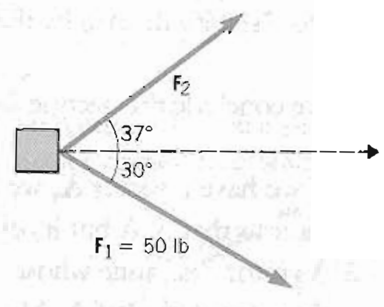


FIGURE 2-8 Example 2-1.

If the object is going to move in the x direction, the resultant force must be in the x direction only, with no component in the y direction. If there is to be no net force in the y direction that could cause the box to move in that direction, then the sum of the positive and negative y forces must be zero. We can express this as

$$\sum F_y = 0$$

$$-25\text{ lb} + 0.6F_2 = 0$$

or

$$F_2 = \frac{25\text{ lb}}{0.6} = 41.7\text{ lb}$$

Question 2 can be answered from the sum of forces in the x direction.

$$\sum F_x = 43.3\text{ lb} + 0.8F_2$$

$$= 43.3\text{ lb} + 0.8 \times 41.7\text{ lb}$$

$$= 76.7\text{ lb}$$

Therefore, we conclude that instead of the two forces F_1 and F_2 acting on the box, the same result can be obtained by a single force of 76.7 lb pulling in the x direction.

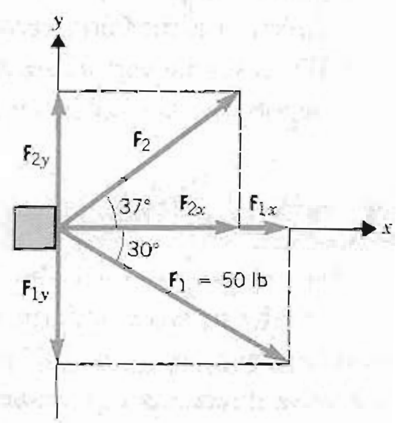


FIGURE 2-9 Example 2-1.

We can now summarize the preceding discussion concerning the addition of vectors by the method of components.

1. Establish a coordinate system. Choose an x-y cartesian coordinate system that is convenient for calculation. For example, we will see in Chapter 4 that when an object slides on an inclined plane, it is convenient to choose the plane as the x axis.
2. Construct each vector with its tail at the origin of this coordinate system.
3. Drop construction lines from the head of each vector perpendicular to the x and y axes, and note from the laws of trigonometry that the x component of a vector is equal to the magnitude of the vector multiplied by the cosine of the angle that the vector makes with the x axis. Similarly, the y component of a vector equals the magnitude of the vector times the sine of the same angle.
4. Add algebraically the individual x components and y components of all the vectors to find the x and y components, respectively, of the resultant vector.
5. The square of the resultant equals the sum of the squares of the x and y components of the resultant.
6. To determine the angle of the resultant, it is best to make a sketch of the resultant, showing its x and y components, which will indicate in what quadrant the resultant is. The angle between R and the x axis (either positive or negative direction) is equal to the tangent of the absolute value of the y component of the resultant divided by the absolute value of the x component.

We conclude this section by defining certain rules of vector algebra.

1. If we have a vector **A**, we define a vector $-\mathbf{A}$ as one whose magnitude is the same as that of **A** but its direction is opposite to that of **A**.
2. A vector 2**A** is one whose magnitude is twice that of **A** and whose direction is the same as that of **A**. More generally, when a vector is multiplied or divided by a scalar quantity, we obtain a vector of different magnitude but of the same direction as the initial vector.
3. When several vectors are added, they can be added in any order and thus the distributive law holds; for example, $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

2.3 UNIT VECTORS

It is lengthy to write and say x component, y component, and z component. A shorthand notation is used. Unit Vectors **i**, **j**, and **k** are introduced. These have the respective directions x, y, and z and a magnitude of unity, so they give direction without changing the magnitude. For example, the vector $\mathbf{F}_1$ in Fig. 2-9 that has components 43.3 lb in the x direction and −25 lb in the y direction would simply be written as

$$\mathbf{F}_1 = (43.3\,\mathbf{i} - 25\,\mathbf{j})\ \text{lb}$$

The conventional diagram for three dimensions is shown in Fig. 2-10 with the corresponding unit vectors.

The vector F in Fig. 2-11 has the components shown, which are obtained by the right-triangle method of extending perpendiculars to the axes. This vector would be written as

$$\mathbf{F} = (4\,\mathbf{i} + 8\,\mathbf{j} + 5\,\mathbf{k})\ \text{lb}$$

and its magnitude is obtained from the three-dimensional pythagorean theorem

$$F = \sqrt{(4\ \text{lb})^2 + (8\ \text{lb})^2 + (5\ \text{lb})^2}$$
$$= 10.2\ \text{lb}$$

When desired, the direction can be obtained by standard methods of analytic geometry to obtain its location in space.

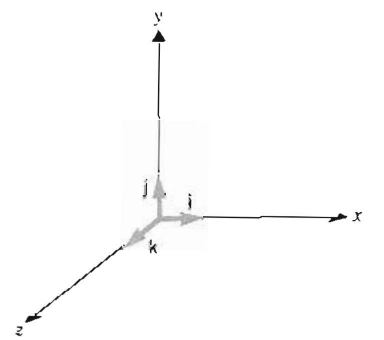


FIGURE 2-10 Unit vectors i, j, k on the three coordinate axes.

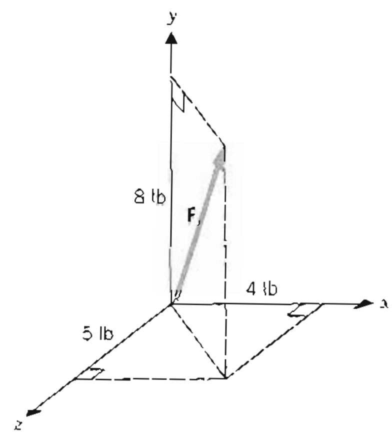


FIGURE 2-11 Components of vector F on the three coordinate axes.

2.4 DOT PRODUCT

Very often in physics we have two vectors with an angle θ between them, and we wish to find the product of their components that lie in the direction of one or the other vector.

Consider Fig. 2-12. If we, for instance, select the A direction, then the component of vector B in that direction is given by dropping a perpendicular (Fig. 2-12a) and noting from the resulting right triangle that the component of B in the A direction is $B \cos\theta$ and the product of this component and vector A is

$$AB\cos\theta$$

If, instead, we had selected the B direction we could equally have dropped a perpendicular from vector A to the line of vector B (Fig. 2-12b) and obtained the identical result. Because there is no specified direction for the resulting product, we define such a product as a scalar. We use the shorthand notation of a dot (·) to represent this type of product, which is referred to as the *dot product*

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \tag{2.1}$$

where on the right side A and B are simply the magnitude of each of the vectors. We will use this dot product in Chapter 5 on work and energy, both of which arise from vector relationships although neither in itself has direction.

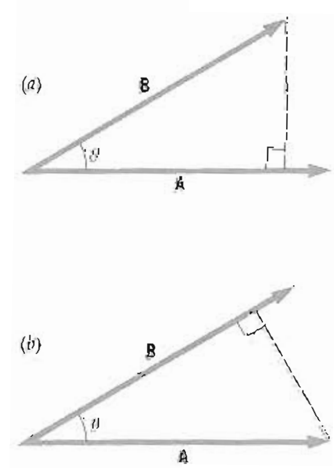


FIGURE 2-12 Geometric representation of two ways of forming a dot product of vectors A and B.

Let us apply our definition of the dot product to the unit vectors i, j, and k.

$$\mathbf{i} \cdot \mathbf{j} = (1)(1)\cos 90^\circ = 0$$

$$\mathbf{i} \cdot \mathbf{k} = (1)(1)\cos 90^\circ = 0$$

$$\mathbf{j} \cdot \mathbf{k} = (1)(1)\cos 90^\circ = 0$$

$$\mathbf{i} \cdot \mathbf{i} = (1)(1)\cos 0^\circ = 1$$

$$\mathbf{j} \cdot \mathbf{j} = (1)(1)\cos 0^\circ = 1$$

$$\mathbf{k} \cdot \mathbf{k} = (1)(1)\cos 0^\circ = 1$$

We see that when a unit vector is dotted with a different unit vector the result is zero, whereas when a unit vector is dotted with itself the result is unity.

EXAMPLE 2-2 Find $\mathbf{A} \cdot \mathbf{B}$ if $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 3\mathbf{j}$.

Solution

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (3\mathbf{i} + 2\mathbf{j}) \cdot (-\mathbf{i} + 3\mathbf{j}) \\ &= 3\mathbf{i} \cdot (-\mathbf{i}) + 3\mathbf{i} \cdot 3\mathbf{j} + 2\mathbf{j} \cdot (-\mathbf{i}) + 2\mathbf{j} \cdot 3\mathbf{j} \\ &= -3 + 0 + 0 + 6 \\ &= 3 \end{aligned}$$

You can verify that $\mathbf{B} \cdot \mathbf{A}$ gives the same answer; therefore the commutative law holds for the dot product of two vectors.

2.5 CROSS PRODUCT

In some topics of physics we often need to define a vector C, whose magnitude is equal to the magnitude of one vector A times the component of a second vector B in the direction perpendicular to A (see Fig. 2-13). Moreover, we want the direction of C to be perpendicular to A and B.

We thus introduce a new type of product called the *cross product* of A and B. If C is the cross product of A and B, we write

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{2.2}$$

By this definition, it is seen in Fig. 2-13 that the magnitude of the vector C is

$$C = AB \sin\theta$$

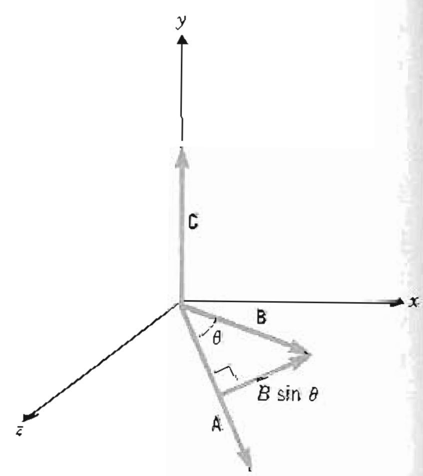


FIGURE 2-13 Geometric representation of the cross product $\mathbf{A} \times \mathbf{B} = \mathbf{C}$, where $C = AB \sin\theta$.

The direction of **C** is perpendicular to both **A** and **B** and consequently perpendicular to the plane containing **A** and **B**. There are obviously two possible directions for a vector perpendicular to a plane. This ambiguity can be removed by using a *right-hand rule*. A simple mnenomic is the following. Consider the two vectors to be two sticks connected by a hinge at the apex of the angle. Mentally place the palm of your right hand against the outside of the first stick to be crossed (in our case **A**) as if to push the two sticks together (see Fig. 2-14). Do this with the thumb extended and the thumb will point in the direction of the vector cross product. The vector $\mathbf{C}' = \mathbf{B} \times \mathbf{A}$ will, from the definition, have the same magnitude as **C**. However, it is clear from Fig. 2-15 that the direction of $\mathbf{C}'$ is opposite to that of **C**; namely,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

Note that this is in contrast to the dot product where $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.

We can examine the resulting directions of cross products by operating on the unit vectors of Fig. 2-10 with the right-hand rule. We find that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

It should be noted in the definition that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ because the angle between a vector and itself is zero and $\sin 0° = 0$.

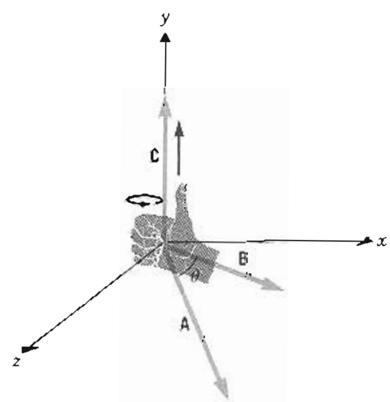


FIGURE 2-14 Right-hand rule for the vector cross product. For $\mathbf{A} \times \mathbf{B}$, curl the fingers of the right hand in a direction such that the fingers seem to push vector **A** toward vector **B**. The direction of the thumb points in the direction of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

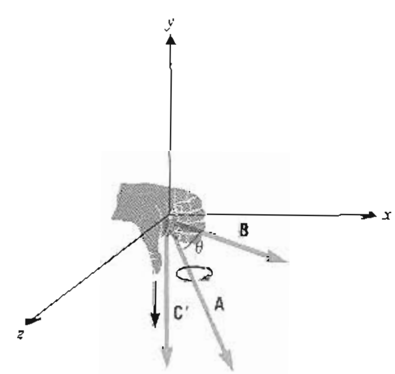


FIGURE 2-15 Right-hand rule for the vector cross product of $\mathbf{B} \times \mathbf{A} = -\mathbf{C}$.

EXAMPLE 2-3 Find $\mathbf{A} \times \mathbf{B}$ if $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 3\mathbf{j}$

Solution

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (3\mathbf{i} + 2\mathbf{j}) \times (-\mathbf{i} + 3\mathbf{j}) \\ &= 3\mathbf{i} \times (-\mathbf{i}) + 3\mathbf{i} \times 3\mathbf{j} + 2\mathbf{j} \times (-\mathbf{i}) + 2\mathbf{j} \times 3\mathbf{j} \\ &= 0 + 9\mathbf{k} + 2\mathbf{k} + 0 \\ &= 11\mathbf{k}\end{aligned}$$

PROBLEMS

2.1 What are the x and y components of the following vector displacements? (a) 2 m at 20°? (b) 3 m at 120°? (c) 4 m at 240°? (d) 2.5 m at 325°? All angles are with respect to the positive x axis.

2.2 A sailboat follows a series of racing buoys. On the first lap it goes 8 mi at 20°, then 10 mi at 40°, then 6 mi at 130°. What distance is it from its starting point, and in what direction must it sail to return?

Answer: 17.8 mi, 230.6°.

2.3 A sailboat sails 7 mi in the direction 37° north of east, then 4 mi in the direction 53° west of north. What is the magnitude and the direction of the final leg that will bring it to the starting point?

2.4 A vector displacement **A** in the x-y plane has an x component of 10 m. The angle between the y axis and the vector **A** is 37°. What is the magnitude of the vector **A**?

2.5 A force of 100 lb acts on an object at an angle of 20° with respect to the x axis, and a force of 300 lb acts at an angle of 60° with respect to the x axis. What single force must be applied at what angle to be the equivalent of these two forces?

Answer: 382 lb, 50.3°.

2.6 In problem 2.5 an additional 200 lb acts at 215°. What single force at what angle will be the equivalent of these three forces?

2.7 Consider the case discussed in Example 2-1, except that $F_1 = F_2 = 50$ lb. The two men exerting these forces on the box ask a small boy to push on the box while they pull it so that it moves only in the x direction with a net force in that direction of 90 lb. With how large a force and in what direction does the boy push on the box?

Answer: 8.4 lb, −37°.

2.8 Express the vectors of problem 2.1 in **i**, **j**, **k** notation.

2.9 Express the vectors of problem 2.2 in **i**, **j**, **k** notation and perform the summation in that notation.

Answer: $11.3\mathbf{i} + 13.8\mathbf{j}$.

2.10 What is the magnitude and the angle of the resultant of vectors **A**, **B**, and **C**, where $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j}$, and $\mathbf{C} = -\mathbf{i} + \mathbf{j}$?

2.11 The sum of three vectors **A**, **B**, and **C** is equal to vector **R**. If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$, and $\mathbf{R} = -2\mathbf{i} + 3\mathbf{j}$, what are the components of vector **C**? Make a sketch of vector **C** on a cartesian system, find its magnitude and the angle it makes with the x axis.

2.12 The resultant of vectors **A**, **B**, and **C** is $2\mathbf{i} + \mathbf{j}$. If $\mathbf{A} = 6\mathbf{i} - 3\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 5\mathbf{j}$, find the components, the magnitude, and the angle of vector **C**.

2.13 Vectors **A** and **B** have magnitudes of 3 m and 4 m, respectively, and are 30° apart. Find $\mathbf{A} \cdot \mathbf{B}$ and the magnitude of $\mathbf{A} \times \mathbf{B}$.

Answer: 10.4 m^2, 6.0 m^2.

2.14 If the vectors **A** and **B** of problem 2.13 are 150° apart, find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$.

2.15 Find the dot and cross products of vectors **A** and **B** of problem 2.13 if they are 0° apart. If they are 180° apart.

2.16 Find the vector $\mathbf{A} \times \mathbf{B}$ if $\mathbf{A} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

2.17 Find the dot product $\mathbf{A} \cdot \mathbf{B}$ if $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = -3\mathbf{j} - 12\mathbf{k}$. What are the magnitudes of vector **A** and vector **B**?

2.18 Use the dot product to find the angle between vectors **A** and **B** in problem 2.17.

2.19 Find the angle between the vectors $\mathbf{A} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 6\mathbf{i} - 3\mathbf{j}$.

Answer: 63.4°.

2.20 What is the angle between the vector $\mathbf{A} = 3\mathbf{i} - 7\mathbf{j}$ and the x-axis?

2.21 Find the angles between the vector $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and the x, y, and z axes, respectively.

Answer: 71.1°, 119.1°, 35.8°.

2.22 Consider a vector $\mathbf{A} = 4\mathbf{i} - 9\mathbf{j}$. Find a vector in the x-y plane that is perpendicular to $\mathbf{A}$.
Answer: $a(2.25\mathbf{i} + \mathbf{j})$, where a is an arbitrary constant.

2.23 A vector $\mathbf{R} = 9\mathbf{i} - 12\mathbf{j}$ can be expressed as a linear combination of two vectors $\mathbf{A}$ and $\mathbf{B}$; namely, $\mathbf{R} = C_1\mathbf{A} + C_2\mathbf{B}$, where C_1 and C_2 are two scalar constants. If $\mathbf{A} = 5\mathbf{i} - 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 12\mathbf{j}$, what are C_1 and C_2?
Answer: $C_1 = 1.68$, $C_2 = -0.58$.

2.24 The resultant $\mathbf{R}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ has half the magnitude of $\mathbf{A}$ and is perpendicular to $\mathbf{B}$. If the magnitude of $\mathbf{A}$ is 5, what is the magnitude of $\mathbf{B}$ and the angle between $\mathbf{A}$ and $\mathbf{B}$?

2.25 Many theorems in geometry can be readily proved by vector algebra. Consider the triangle OAB in Fig. 2-16. Show that the line joining the midpoint of side OA to the midpoint of side AB is parallel to OB and its length is half that of OB. (*Hint*: Make vectors out of OA, OB, AB, CA, AD, and CD and find relations between these vectors)

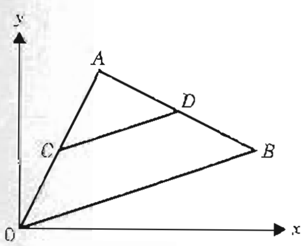


FIGURE 2-16 Problem 2.25.

2.26 Consider the line obtained by joining the origin and the point $x = 5, y = 3$ (see Fig. 2-17). Find the perpendicular distance h from a point P with coordinates $x = 1, y = 7$ to that line.
Answer: 5.49.

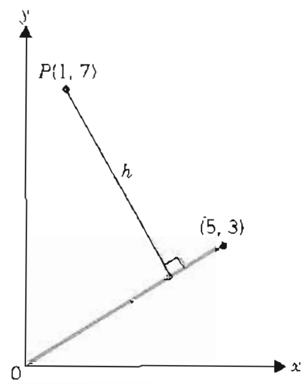


FIGURE 2-17 Problem 2.27.

2.27 The magnitude of vectors $\mathbf{A}$ and $\mathbf{B}$ are 4 and 10, respectively. The magnitude of the resultant $\mathbf{R}$ is 12. What is the angle between $\mathbf{A}$ and $\mathbf{B}$?

2.28 What is the area of the triangle formed by joining the following three points: $x = 0$ m, $y = 0$ m; $x = 3$ m, $y = 4$ m; $x = 7$ m, $y = 2$ m. Recall that the area of a triangle is Area $= 1/2$ base $\times$ height. (*Hint*: Make vectors out of two of the sides of the triangle and consider the magnitude of the cross product of those two vectors.)
Answer: 11 m^2.

3.1 INTRODUCTION

In this chapter we introduce certain vector quantities—position, displacement, velocity and acceleration—used to describe the motion of a body. We define these quantities and discuss the mathematical relations between them. We then derive specific functional relations between them and time (a scalar quantity) for the case where the object moves in a straight line with constant acceleration. The chapter concludes with a discussion of projectile motion, one of the simplest types of two-dimensional motion.

3.2 SPEED AND VELOCITY

Two words in English, "speed" and "velocity," are used interchangeably to indicate how fast a body is moving. In physics we make a distinction between them. The word "speed" is defined as a scalar quantity and "velocity" is a vector quantity. Thus, the average speed (where average will be represented by a bar on top of the quantity involved) is the distance traveled in any direction, Δs, divided by the time Δt, or

$$\overline{\text{speed}} = \frac{\Delta s}{\Delta t} \tag{3.1}$$

where

$$\Delta(\text{anything}) = \text{final value} - \text{initial value}$$

Velocity is defined differently. Consider a particle moving in space. Let the particle be at point P in Fig. 3-1 at some initial time t_0 and at point P' some later time t_f. The initial position of the particle can be specified by a *position vector* $\mathbf{r}_0$ obtained by drawing an arrow from the origin of the coordinate system to point P. Similarly, the position at the later time is specified by a second position vector $\mathbf{r}_f$ that results when an arrow is drawn from the origin to point P'. The position at any other point in the motion is specified by a corresponding position vector $\mathbf{r}$. We can now define the ***displacement vector*** $\Delta\mathbf{r}$ as the vector difference between the final and the initial position vectors, namely, $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_0$ (see Fig. 3-1). Correspondingly, we define the ***average velocity*** $\overline{\mathbf{v}}$ as the ratio of the displacement vector to the time taken for the displacement to occur, namely,

$$\overline{\mathbf{v}} = \frac{\mathbf{r}_f - \mathbf{r}_0}{t_f - t_0} = \frac{\Delta\mathbf{r}}{\Delta t} \tag{3.2}$$

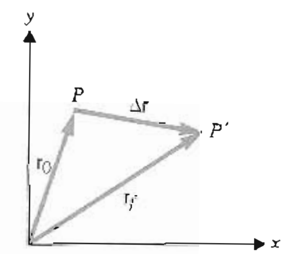


FIGURE 3-1 The displacement vector $\Delta\mathbf{r}$ is obtained by drawing an arrow from the initial position vector $\mathbf{r}_0$ to the final position vector $\mathbf{r}_f$.

The distinction between speed and velocity is difficult to grasp at first, but it is extremely important. Consider the walk taken in Fig. 2-1. Suppose it took 1 h. Then, by definition, Eq 3.1, the average speed of walking was $\Delta s/\Delta t = (4.0\,\text{mi} + 5.0\,\text{mi})/1\,\text{h} = 9\,\text{mi/h}$, whereas the average velocity was $\Delta\mathbf{r}/\Delta t = 6.4\,\text{mi}/1\,\text{h} =$

6.4 mi/h in the direction 39° north of east. Consider a more extreme example. Suppose a race car is traveling around a circular track of 1-mi diameter and its speedometer reads 100 mi/h. This is the speed. The time taken to reach any point is, from Eq. 3.1, $\Delta t = \Delta s/\overline{\text{speed}}$. Because the track length is $\pi \times$ diameter $= 3.14$ mi, the time to complete one circuit is

$$\Delta t = \frac{3.14\,\text{mi}}{100\,\text{mi/h}} = 3.14 \times 10^{-2}\,\text{h}$$

and the time to go halfway around is 1.57×10^{-2} h. However, the car's average velocity by the definition of Eq. 3.2 depends on its position. When the car has gone halfway around, say from the western-most to eastern-most position on the track, then the magnitude of the vector displacement from the starting point is the diameter or 1 mi. Hence, its average velocity to that point is

$$\overline{\mathbf{v}} = \frac{1\,\text{mi}}{1.57 \times 10^{-2}\,\text{h}} = 63.7\,\text{mi/h in the east direction.}$$

In one complete circuit, as the car passes the starting point its vector displacement is zero and hence

$$\overline{\mathbf{v}} = \frac{0\,\text{mi}}{3.14 \times 10^{-2}\,\text{h}} = 0\,\text{mi/h}$$

This seeming contradiction has occurred because we have taken large displacements for $\Delta\mathbf{r}$. If we shrink the displacement to a minute amount by taking the limit

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \tag{3.3}$$

then the magnitude of the velocity, which is now called the *instantaneous velocity*, at any point on the track will equal the speed. This can be seen in Fig. 3-2, where we notice that as Δs becomes smaller, the difference between Δs and the corresponding $\Delta\mathbf{r}$ decreases. We should also note that in the limit where $\Delta\mathbf{r}$ becomes infinitesimally small, it becomes tangential to the path, and therefore the direction of the instantaneous velocity is the tangent to the path. Thus, while the magnitude

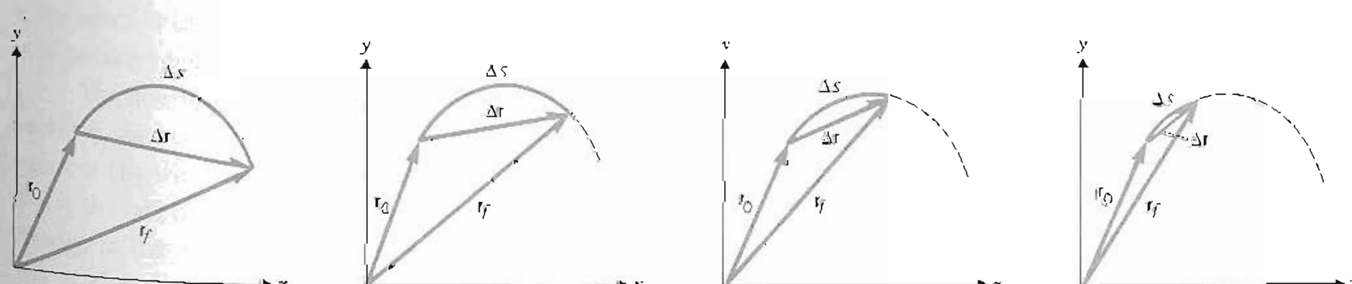


FIGURE 3-2 A curved path of a car traveling clockwise. Δs is the distance traveled by the car, and $\Delta\mathbf{r}$ is the displacement vector between the position of the car $\mathbf{r}_f$ at some instant and the position $\mathbf{r}_0$ at the initial time. As the distance Δs becomes smaller, its value approaches $\Delta\mathbf{r}$.

of the instantaneous velocity remains equal to the speed, the direction part of the instantaneous velocity is changing. Velocity is a vector because it is equal to a vector displacement divided by time, which is scalar, and the division of a vector by a scalar does not remove the vector property. We should note that by definition, Eq. 3.3, the instantaneous velocity **v** is the first derivative of the position vector with respect to time. It should also be pointed out that because Eq. 3.3 is a vector equation, it holds for each of the cartesian components of the vectors **v** and **r**, namely,

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}$$

where v_x, v_y, and v_z are the cartesian components of **v** and x, y, and z are those of **r**.

3.3 ACCELERATION

In the preceding section we introduced a convention in which Δ is a measurable change between a final value and an initial value and d is used for an infinitesimally small change.

If there is a velocity change $\Delta\mathbf{v}$ in a certain time Δt, we define the average acceleration as

$$\bar{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t} \tag{3.4}$$

or

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_f - \mathbf{v}_0}{t_f - t_0}$$

where the subscripts f and 0 represent final and initial values, respectively. Usually in a problem we start our stopwatch at $t_0 = 0$, so the elapsed time is simply t_f and we drop the subscript f. We may define an instantaneous acceleration as

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \tag{3.5}$$

which is the first derivative of **v** with respect to time. Substituting Eq. 3.3 for **v**, we write

$$\mathbf{a} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2} \tag{3.6}$$

which is the second derivative of **r** with respect to time.

EXAMPLE 3-1 The position of a body on the x axis varies as a function of time according to the following equation

$$x(\text{meters}) = (3t + 2t^2)\text{m}$$

Find its velocity and acceleration when $t = 3$ sec.

Solution Because the body moves in a straight line, r = x. From Eq. 3.3

$$v = \frac{dx}{dt} = \frac{d}{dt}(3t + 2t^2) = (3 + 4t)\ \text{m/sec}$$

The velocity of the body at $t = 3$ sec is therefore

$$v(t = 3\ \text{sec}) = 3 + 4 \times 3 = 15\ \text{m/sec}$$

From Eq. 3.5,

$$a = \frac{dv}{dt} = \frac{d}{dt}(3 + 4t) = 4\ \text{m/sec}^2$$

Notice that a is a constant, and therefore $a(t = 3\ \text{sec}) = 4\ \text{m/sec}^2$.

3.4 LINEAR MOTION

Because displacement, velocity, and acceleration are vectors, we may treat them by the method of cartesian components introduced in Chapter 2. First, let us consider motion only in the direction of a single component, for example, the x direction, that is, motion in a straight line.

If we start timing an object moving in the x direction when it starts from or is passing the $x = 0$ point, we may write Eq. 3.2 as

$$\overline{v}_x = \frac{x - 0}{t - 0}$$

or

$$x = \overline{v}_x t \tag{3.7}$$

Because in this section we will be talking about motion in one direction, we will drop the subscript x from the velocity.

Equation 3.7 results from the definition of average velocity; thus it holds in all cases whether or not the acceleration is constant. In the remainder of this chapter, we will consider only *constant acceleration*.

The derivative of a variable, for example, the velocity v, with respect to a second variable, for example, time t, represents the instantaneous *rate of change* of the first variable (v) with respect to the second (t). Thus, the acceleration as defined in Eq. 3.5 is the rate of change of the velocity with time. If the acceleration is constant, the change in the velocity during the first, second, third, and all succeeding seconds of the motion will be the same and equal to the acceleration a. Thus, if the motion lasts t seconds, the change in the velocity $\Delta v = v - v_0 = at$, where v is the final velocity and v_0 is the initial velocity. We can rewrite this result as

If velocity v is plotted against time t, it is seen that Eq. 3.8 is a straight line, as indicated in Fig. 3-3. The slope of this line is the constant acceleration a.

With a little bit of thought, we can obtain another important relation. When the velocity increases at a constant rate as in Eq. 3.8 and Fig. 3-3, the average velocity is one half the sum of the initial velocity v_0 and the final velocity v, namely,

$$\bar{v} = \frac{v + v_0}{2} \tag{3.9}$$

and Eq. 3.7 becomes

$$x = \frac{v + v_0}{2} t \tag{3.10}$$

The three equations (3.7), (3.8), and (3.9) define linear motion for constant acceleration. Often, however, at least two of these, and sometimes all three, must be used to solve a problem. It is convenient, therefore, to combine these three equations to obtain two auxiliary ones and have all five available (thereby avoiding the necessity of solving simultaneous equations). We obtain the auxiliary equations in the following way.

From Eq. 3.8 write

$$t = \frac{v - v_0}{a}$$

and substituting it into Eq. 3.10 we obtain

$$x = \frac{(v + v_0)}{2} \frac{(v - v_0)}{a}$$

and

$$v^2 - v_0^2 = 2ax \tag{3.11}$$

If we substitute Eq. 3.8 for v in Eq. 3.10, we obtain

$$x = \frac{v_0 + at + v_0}{2} t$$

and

$$x = v_0 t + \frac{1}{2} a t^2 \tag{3.12}$$

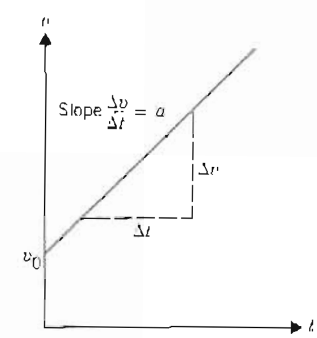


FIGURE 3-3 Plot of v versus t for constant acceleration.

We may derive these equations more formally by integration. By definition

$$a = \frac{dv}{dt} \tag{3.5}$$

Rearranging terms and integrating, we write

$$\int_{v_0}^{v} dv = \int_0^t a \, dt$$

This equation holds in general whether or not acceleration is a constant. In the present case, acceleration is taken as constant, so a can be taken out of the integral and we write

$$\int_{v_0}^{v} dv = a \int_0^t dt$$

This integrates to

$$v - v_0 = at$$

and

$$v = v_0 + at \tag{3.8}$$

From the definition

$$v = \frac{dx}{dt} \tag{3.3}$$

$$\int_{x_0}^{x} dx = \int_0^t v\, dt$$

Substitute Eq. 3.8 for v

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + at)\, dt = v_0 \int_0^t dt + a \int_0^t t\, dt$$

$$x - x_0 = v_0 t + \frac{1}{2} a t^2 \tag{3.12}$$

Note that in this formulation of Eq. 3.12 we have not required that $x = 0$ at $t = 0$ as in the previous algebraic derivations.

We may use the chain rule to write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{3.13}$$

or

$$\int_{v_0}^{v} v\, dv = a \int_{x_0}^{x} dx$$

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

$$v^2 - v_0^2 = 2a(x - x_0) \tag{3.11}$$

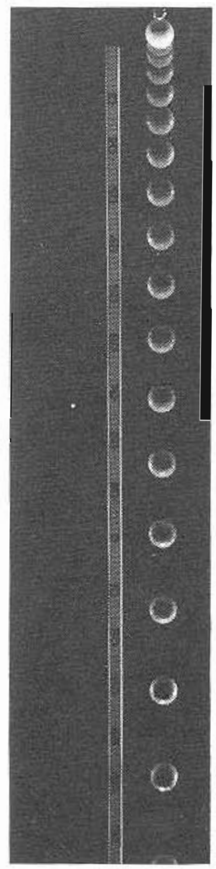
Equations 3.7 through 3.12 have been derived for motion in the x direction. Similar equations can simply be written for motion in the y and z directions when the components of the acceleration in these directions are also constant.

There is one important thing to be noted here. In the solution of motion problems we must assign vector directions. Suppose we observe a boy throwing a ball as we look through a transparent piece of graph paper and draw lines of motion and displacement. We could lie on our side or stand on our head and draw the lines without having any effect on the boy and his ball. Therefore, choosing a particular coordinate system is a matter of personal convenience. The upward direction could be the positive y direction or the negative y direction, or even a direction at an angle on the graph paper, although we will always try to choose a system that will minimize the calculational steps. It is important to note that once we choose a coordinate system, all parameters have their vector direction controlled by it. If we choose the positive y direction as up and the boy throws the ball straight up, then the vector displacement from the ground to its highest position is positive. During its upward travel, because velocity is the displacement divided by the scalar time, it too is positive. The only motion is in the y direction, so we therefore use y, v_y, and a_y in the equations previously derived. Thus, Eq. 3.8 is

$$v_y = v_{0y} + a_y t \qquad (3.8')$$

We observe that in throwing the ball upward the largest value for the magnitude of the y velocity occurs as it leaves the boy's hand; then the ball begins to slow down until the upward velocity is zero. The only way that this can occur is if a_y in Eq. 3.8′ is negative. Now a_y is the acceleration caused by the force of gravity acting on the ball, and we will see in the next chapter that because the force of gravity is downward then the corresponding acceleration must also be downward. *The acceleration caused by gravity is usually written as the symbol g and has the approximate sea-level value $g = 9.8$ m/sec^2.*

When solving problems, the best approach is to tabulate what is known and what is to be found and select the appropriate equation.



Multiflash photograph of a falling ball. Note the increase in the distance traveled by the ball between flashes as it falls. This reflects an increase in the velocity of the ball caused by the downward-directed gravitational acceleration.

EXAMPLE 3-2 A boy throws a ball upward with an initial velocity of 12 m/sec. How high does it go?

Solution We choose the starting point as the origin and the upward direction as positive. Because velocity is a vector displacement divided by time, upward velocity is also positive. The force of gravity is in the negative y direction, so the sign of the acceleration is therefore negative. First list what is known and what is to be found

$$v_{0y} = 12 \text{ m/sec}, \quad v_y = 0 \text{ (at its highest point)}, \quad a_y = g = -9.8 \text{ m/sec}^2$$

$$y = ?$$

We select the y equivalent of Eq. 3.11 because all the quantities in that equation are known except y, the quantity that we want to find

$$v_y^2 - v_{0y}^2 = 2a_y y$$

Solving for y, we write

$$y = \frac{v_y^2 - v_{0y}^2}{2a_y}$$

Substituting the numerical values for the quantities in the equation,

$$y = \frac{0 - (12\,\text{m/sec})^2}{2(-9.8\,\text{m/sec}^2)}$$

$$= 7.3\,\text{m}$$

EXAMPLE 3-3 A boy throws a ball upward with an initial velocity of 12 m/sec and catches it when it returns. How long was it in the air?

Solution As in the previous example, we choose the starting point as the origin and the upward direction as positive.

$v_{0y} = 12\,\text{m/sec}$, $a_y = -9.8\,\text{m/sec}^2$, $y = 0$ (vector displacement is zero because it returns to his hand), $t = ?$

Select Eq. 3.12

$$y = v_{0y}t + \frac{1}{2}a_y t^2$$

Using the fact that $y = 0$, Eq. 3.12 becomes

$$0 = v_{0y}t + \frac{1}{2}a_y t^2$$

We see immediately that if we divide both sides of the equation by t, we obtain

$$0 = v_{0y} + \frac{1}{2}a_y t$$

and

$$t = -\frac{2\,v_{0y}}{a_y}$$

$$= -\frac{2 \times 12\,\text{m/sec}}{-9.8\,\text{m/sec}^2}$$

$$= 2.45\,\text{sec}$$

In this example, the ball returned to its starting point, so the displacement y was zero. This simplified Eq. 3.12, because dividing both sides by t linearized the equation. If the ball had landed on a roof, then the left side of Eq. 3.12 would not be zero and the equation to be solved would be quadratic.

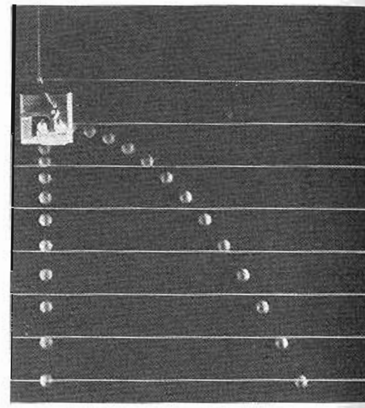
3.5 PROJECTILE MOTION

We have treated motion in one dimension in the preceding section. Suppose we have a smooth, frictionless wall in the x-y plane. If we set an object in motion along the wall, we find experimentally that the object is accelerated downward in the y-vector direction but that there is no acceleration in the x-vector (horizontal) direction. That is, the object moves in the x direction with its constant initial x velocity but its y velocity is increasing downward owing to the acceleration of gravity. If we now perform the same experiment with an imaginary wall, we have what is called *projectile motion*. The characteristic in the coordinate system in which we are working is that the x and y motions and velocities are at right angles to each other and that there is an acceleration only in the y direction, a_y; there is no acceleration in the x direction. The equations of motion in the x and y directions are therefore

$$x = v_{0x}t$$

$$y = v_{0y}t + \frac{1}{2}a_y t^2$$

In view of the foregoing, projectile problems are treated as two separate linear motion problems, one in the x direction and another in the y direction, with only time as the common element.



Multiflash photograph of two falling balls: One released from rest and the other launched with an initial horizontal velocity. The vertical position of the two balls at any given time (time of the flashes) is the same for both, indicating that the vertical motion is unaffected by the initial horizontal velocity given to the second ball.

EXAMPLE 3-4 A ball moving at 2 m/sec rolls off of a 1-m-high table, Fig. 3-4. How far horizontally from the edge of the table does it land?

Solution The ball will continue moving in the x direction for as long as it is in the air. We can use Eq. 3.7 to determine the x coordinate of the ball as it lands

$$x_f = \overline{v}_x t_f$$

where t_f is the time that the ball is in the air. Note here that, because the velocity in the x direction is unchanged, $\overline{v}_x = v_{0x}$ and therefore

$$x_f = 2\,\text{m/sec}\; t_f$$

t_f is the time when the y coordinate of the ball becomes -1 m. We have

$$y_f = -1\,\text{m},\quad v_{0y} = 0,\quad a_y = -9.8\,\text{m/sec}^2,\quad t_f = ?$$

Using Eq. 3.12,

$$y = v_{0y}t + \frac{1}{2}a_y t^2$$

Because $v_{0y} = 0$, this equation becomes

$$y = \frac{1}{2}a_y t^2$$

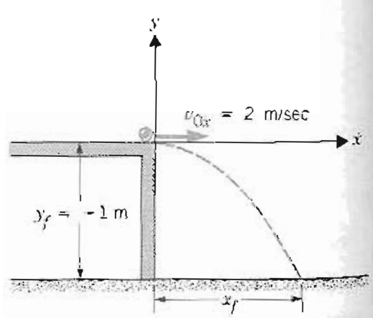


FIGURE 3-4 The edge of the table is chosen as the origin of the x-y coordinate system.

and

$$t = \pm\sqrt{\frac{2y}{a_y}}$$

$$t_f = \pm\sqrt{\frac{2(-1\ \text{m})}{-9.8\ \text{m/sec}^2}}$$

$$= \pm\, 0.45\ \text{sec}$$

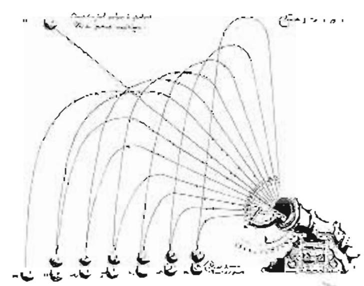
and because in deriving Eqs. 3.9 through 3.12 we chose $t = 0$ as the initial time, only the positive root is acceptable, therefore,

$$x_f = 2\ \text{m/sec} \times 0.45\ \text{sec}$$

$$= 0.9\ \text{m}$$

Let us examine a specific case of projectile motion along level ground. We will find the general formula for the distance that a person can throw a ball or that a gun can fire a projectile. The variables are shown in Fig. 3-5a and the initial velocity components in Fig. 3-5*b*. The distance x that the projectile travels just before it strikes the ground is

$$x_f = \bar{v}_x t_f = v_{0x} t_f \quad (\text{because } \bar{v}_x = v_{0x})$$



Cannonball trajectories for various launching angles as conceived by Diego Ufano in 1621.

From Fig. 3-5*b*

$$v_{0x} = v_0 \cos\theta$$

and therefore

$$x_f = v_0 \cos\theta\, t_f$$

We determine the time in the air from the y-direction problem when the projectile is thrown upward with an initial velocity of $v_{0y} = v_0 \sin\theta$ and is acted on by the acceleration of gravity, $a_y = g = -9.8\ \text{m/sec}^2$. At the end of its trajectory the vector displacement is $y = 0$, so we may write

$$y_f = 0,\quad v_{0y} = v_0 \sin\theta,\quad a_y = -9.8\ \text{m/sec}^2,\quad t_f = ?$$

using

$$y = v_{0y}t + \frac{1}{2}a_y t^2 \tag{3.14}$$

$$0 = v_0 \sin\theta\, t_f - \frac{1}{2}g t_f^2$$

$$t_f = \frac{2}{g} v_0 \sin\theta$$

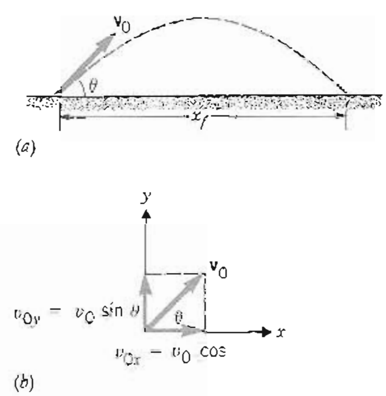


FIGURE 3-5 (*a*) Projectile motion on level ground with v_0 and θ the initial velocity and angle, respectively. (*b*) The x and y

Substitute this for t_f in the equation for x-direction motion

$$x_f = \frac{v_0^2}{g} 2 \sin\theta \cos\theta$$

Substitute the trigonometric relation $2\sin\theta\cos\theta = \sin 2\theta$ and obtain

$$x_f = \frac{v_0^2}{g} \sin 2\theta \qquad (3.15)$$

We can readily find from Eq. 3.15 the angle at which the projectile should be thrown (or fired) to achieve a maximum distance in the x direction for a fixed value of v_0. The only variable is the angle in the term $\sin 2\theta$, and the sine has a maximum value of unity when the argument is 90°. Therefore, x_f of Eq. 3.15 is maximum when $2\theta = 90°$ or $\theta = 45°$.

Note that Eq. 3.15 is valid only when the projectile returns to the starting level, because we set $y_f = 0$ in Eq. 3.14. If it returns to some other level, then the quadratic equation in time must be solved (see problems 3.16 and 3.17).

EXAMPLE 3-5 A boy stands on the edge of a roof 10 m above the ground and throws a ball with a velocity of 15 m/sec at an angle of 37° above the horizontal. How far from the building does it land? See Fig. 3-6.

Solution Let us choose the edge of the roof as the origin of the coordinate system. There is no acceleration in the x direction, so we may simply write the following for x distance

$$x_f = \bar{v}_x t_f = v_{0x} t_f$$

$$v_{0x} = v_0 \cos 37° = 15\ \text{m/sec} \times 0.8 = 12\ \text{m/sec}$$

$$x_f = 12\ \text{m/sec}\ t_f$$

We now need to find the time in the air, which is a motion problem in the y direction only.

$$y_f = -10\ \text{m}, \quad v_{0y} = 15\ \text{m/sec} \sin 37° = 15\ \text{m/sec} \times 0.6 = 9\ \text{m/sec}$$

$$a_y = -9.8\ \text{m/sec}^2, \quad t_f = ?$$

We use Eq. 3.12

$$y = v_{0y} t + \frac{1}{2} a_y t^2$$

If $t = t_f$ when $y = y_f = -10$ m, this equation can be written as

$$\frac{1}{2} a_y t_f^2 + v_{0y} t_f - y_f = 0$$

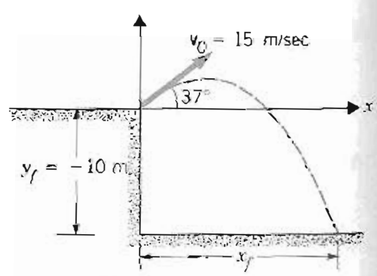


FIGURE 3-6 Example 3-5.

Solving this quadratic equation for t_f

$$t_f = \frac{-v_{0y} \pm \sqrt{v_{0y^2} - (4)\left(\frac{1}{2}a_y\right)(-y_f)}}{(2)\left(\frac{1}{2}a_y\right)}$$

$$= \frac{-v_{0y} \pm \sqrt{v_{0y^2} + 2a_y y_f}}{a_y}$$

Substituting the numerical values for y_f, v_{0y}, and a_y

$$t_f = \frac{-(9\,\text{m/sec}) \pm \sqrt{(9\,\text{m/sec})^2 + (2)(-9.8\,\text{m/sec}^2)(-10\,\text{m})}}{-9.8\,\text{m/sec}^2}$$

$$t_f = 2.6\,\text{sec}, \quad -0.78\,\text{sec}$$

Because we have started timing when the ball is thrown, the negative time solution is rejected because it has no physical meaning to this problem. Substitute the positive time of 2.6 sec into the equation of motion in the x direction and obtain

$$x_f = 12\,\text{m/sec} \times 2.6\,\text{sec} = 31.2\,\text{m}$$

It will be instructive to find the magnitude of the velocity and the angle at which the ball strikes the ground. This is obtained from the components of the velocity just before it hits, as shown in Fig. 3-7. We see from the vector component method that the ball's vector velocity just before striking the ground is given by the final value of its components, v_{fx} and v_{fy}. Because we have a right triangle, we may use the pythagorean theorem

$$v_f^2 = v_{fx}^2 + v_{fy}^2$$

and the angle θ at which it strikes the ground is

$$\theta = \arctan \frac{v_{fy}}{v_{fx}}$$

$v_{fx} = 12$ m/sec because it is unchanged during the ball's flight. We must therefore find v_{fy}. We obtain this from Eq. 3.8 in the y direction.

$$v_{0y} = 9\,\text{m/sec}, \quad t_f = 2.6\,\text{sec}, \quad a_y = -9.8\,\text{m/sec}^2, \quad v_{fy} = ?$$

$$v_{fy} = v_{0y} + a_y t_f$$

$$v_{fy} = 9\,\text{m/sec} - 9.8\,\text{m/sec}^2 \times 2.6\,\text{sec}$$

$$v_{fy} = -16.5\,\text{m/sec}$$

Then

$$v_f = \sqrt{(12\,\text{m/sec})^2 + (-16.5\,\text{m/sec})^2} = 20.4\,\text{m/sec}$$

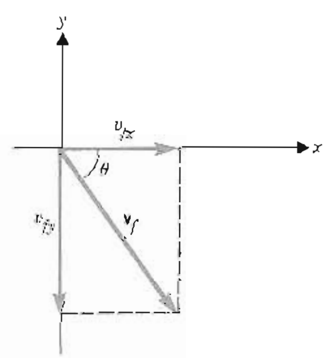


FIGURE 3-7 Example 3-5. Diagram to calculate the angle at which the projectile strikes the ground.

$$\theta = \arctan \frac{|-16.5\ \text{m/sec}|}{12\ \text{m/sec}} = 54^\circ$$

where θ is the angle indicated in Fig. 3-7.

PROBLEMS

3.1 A student drives to college 15 km away from home in half an hour. After classes, he returns home in 20 min. Find (a) the average speed on his way to college, (b) the average speed for the round trip, (c) his average velocity for the entire trip.

3.2 The position of a particle moving along the x axis is given by $x = 3 + 17t - 5t^2$, where x is in meters and t is in seconds. (a) What is the position of the particle at $t = 1$, 2, and 3 sec? (b) At what time does the particle return to the origin? (c) What is the instantaneous velocity at $t = 1$, 2, and 3 sec? (d) At what time is the instantaneous velocity of the particle zero? (e) What is the velocity of the particle as it passes through the origin? (f) What is the acceleration of the particle as it passes the origin?

3.3 The position of a particle moving in a straight line is given by $x = 5 + 2t + 4t^2 - t^3$, where x is in meters. (a) Find an expression for the instantaneous velocity as a function of time. (b) Find the position of the particle at $t = 0, 1, 0.1$, and 0.01 sec. (c) What is the average velocity between $t = 0$ sec and $t = 1$ sec, between $t = 0$ sec and $t = 0.1$ sec, and between $t = 0$ sec and $t = 0.01$ sec? (d) What is the instantaneous velocity at $t = 0$ sec? (e) What conclusion do you draw from the answers in (c) and (d)?

3.4 A car is driving east at 60 km/h, it then makes a turn and travels north at 50 km/h. If it takes 2 sec to make the turn, what is the average acceleration of the car over this 2 second interval?

Answer: 10.85 m/sec^2, directed 39.8° north of west.

3.5 Consider the particle of problem 3.3. (a) Find an expression for the acceleration of the particle as a function of time. (b) What is the instantaneous velocity of the particle at $t = 0$, 1, 0.1, and 0.01 sec? (c) What is the average acceleration between $t = 0$ sec and $t = 1$ sec, between $t = 0$ sec and $t = 0.1$ sec, between $t = 0$ sec and $t = 0.01$ sec? (d) What is the instantaneous acceleration at $t = 0$ sec? (e) What conclusion can you draw from the answers in (c) and (d)?

3.6 A car starts from rest and accelerates uniformly to a speed of 25 m/sec in 8 sec. (a) What is the acceleration? (b) How far did it travel in the 8 sec?

Answer: (a) 3.13 m/sec^2, (b) 100 m.

3.7 A rocket starting from rest rises to a height of 20,000 m in 60 sec. (a) What was the average velocity of the rocket? (b) Assuming that the acceleration was constant, what was the acceleration of the rocket? (c) What was the velocity and the height of the rocket after 30 sec?

3.8 A boy stands on the edge of a building 10 m above the ground and throws a ball upward with an initial velocity of 12 m/sec. It misses the roof on the way down and falls to the ground. Find how long the ball was in the air and its velocity just before it strikes the ground. (*Hint*: take $y = 0$ at $t = 0$ and y final as -10 m).

Answer: 3.11 sec, -18.44 m/sec.

3.9 A car moving at 25 m/sec strikes a tree, and the tree is seen to dent the front by 0.5 m. Assume that the deceleration of the car was constant. Find the deceleration and time it took the car to stop.

3.10 A car moving with constant acceleration covers a distance of 50 m between two points in 5 sec. Its velocity as it passes the second point is 16 m/sec. (a) What is its acceleration? (b) What was its velocity as it passed the first point?

Answer: (a) 2.4 m/sec^2, (b) 4.0 m/sec.

3.11 A ball is dropped from the roof of a building. It is observed to take 0.2 sec to pass by a window 2 m high. How far is the top of the window from the roof?

Answer: 4.15 m.

3.12 A ball is dropped from a bridge 60 m above the surface of the water. One second later, a second ball is thrown down with an initial velocity v_0. Both balls strike the water at the same time. (a) How long were the balls in the air? (b) What was the initial velocity of the second ball? (c) What were the velocities of the balls as they struck the water?

Answer: (a) 3.50 sec, 2.50 sec, (b) −11.76 m/sec, (c) −34.30 m/sec, −36.25 m/sec.

3.13 A motorcycle is waiting at an intersection. As the light turns green it starts with an acceleration of 20 m/sec^2. At that same moment a car, moving with constant velocity of 120 m/sec overtakes and passes the motorcycle. (a) How far from the traffic light will the motorcycle overtake the car? (b) What is the velocity of the motorcycle at that point?

3.14 A girl drops a flowerpot from a window 50 m above the ground. At the same instant a boy directly under the flowerpot throws a stone with an upward velocity of 30 m/sec. (a) How far above the ground will the stone hit the pot? (b) How long after the flowerpot was dropped does the hit take place? (c) What is the minimum velocity with which the stone must be thrown for the hit to occur?

Answer: (a) 36.4 m, (b) 1.67 sec, (c) 15.65 m/sec.

3.15 An electron is set in motion horizontally with a velocity $v_x = 4 \times 10^6$ m/sec. How far will it fall while traveling a horizontal distance of 10 m?

3.16 A boy standing on the ground throws a ball at an angle of 37° above the horizontal with a velocity of 15 m/sec. It lands on the edge of a flat roof of a building 3 m high. How far horizontally from the boy does it strike the roof?

Answer: 16.86 m.

3.17 A boy standing on the ground throws a ball at an angle of 37° above the horizontal with a velocity of 15 m/sec. It strikes the wall of a building 16 m away. How high above the ground is the point at which the ball strikes the building?

Answer: 3.32 m.

3.18 An artillery gunner wishes to have a projectile land at a point on level ground 20,000 m away from the gun. If the muzzle velocity is 500 m/sec and the muzzle is assumed to be at ground level, at what angle above the horizontal should the gun be aimed?

3.19 If the gun of problem 3.18 is on a hill 30 m high and the same angle of elevation is used, how far beyond the target will the projectile land?

3.20 A batter at home plate hits a baseball 1 m above the ground. The ball leaves the bat in the direction of an outfielder with a velocity of 30 m/sec at an angle of 30° above the horizontal. Half a second after the ball is hit, the outfielder 100 m away from home plate runs to catch the ball. How fast must he run to catch the ball just before it hits the ground?

Answer: 7.16 m/sec.

4.1 INTRODUCTION

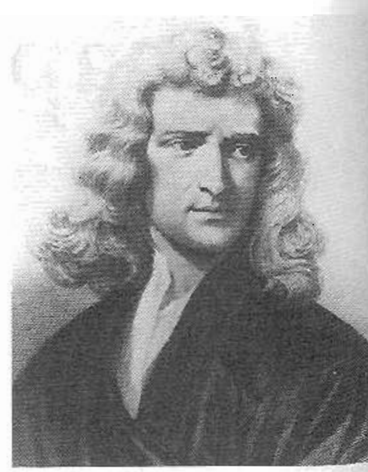
In this chapter we will consider Newton's three laws of motion. Although when first propounded they were postulates, they have since been verified by experiment in so many ways that they are now considered Laws of Nature. There is one consistent word in these three laws and that is "body." We sometimes speak of this as the *newtonian body*. Notice that body is singular. In a given physical situation we must first define the newtonian body, which may often be a mathematical point, in which case it is called a *particle*. If the situation has two bodies, then Newton's laws must be applied separately to each. Often we solve complicated systems of solids on a computer that can remember and vary 10,000 atomic positions in a solid. We require an equal number of applications of Newton's laws, one for each, although the complexity of the solution of that number of simultaneous equations is often reducible by symmetry of behavior.

In addition to using the basic terms of Chapter 1—length, mass, and time—we will discuss the term *force*, which we have used a bit freely. A precautionary word might be said of these.

Length, at least on earth, can be measured by adopting a standard such as the length of the king's foot, or the length of a standard bar of metal, and comparing other lengths with it.

Time can be measured by divisions of the motion of the earth either in rotation about its own axis or in revolution about the sun. Again, a standard has been established by our environment. An early recorded question about what time really means is found in a discussion by St. Augustine in his *Confessions* around 400 A.D. "For so it is O Lord, my God, I measure it, but what it is I measure I do not know."

Mass is an even more obscure property. Newton first referred to it as *inertial* mass, that property of a body which resists being set in motion if a force is applied. But what is *force*? It is that entity which under certain conditions, to be discussed shortly, can change the state of motion of a mass when it acts on it. And thus we find ourselves in a circular argument. We may define mass if force is known, or we may define force if mass is known. The customary approach is to start with mass and define force through the motion it causes on mass. This enables us to use a combination of dimensions—length, mass, and time for force. So if we choose an arbitrary object, and agree that it be the standard of mass (the kilogram was selected), we will be able to evaluate the mass of any other body by means of Newton's second law. We will shortly see how this is done.



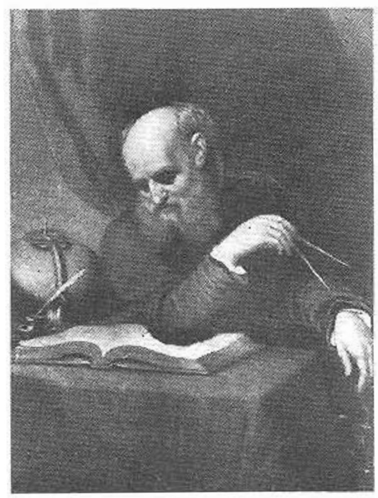
Isaac Newton (1642–1727).

4.2 NEWTON'S LAWS

We will not state the three laws exactly as Newton did; instead we will use modern English so that we may discuss them with no misunderstanding.

- *First Law*: Every body of matter continues in a state of rest or moves with constant velocity in a straight line unless compelled by a force to change that state.
- *Second Law*: When net unbalanced forces act on a body, they will produce a change in the *momentum* (this concept will be defined shortly) of that body proportional to the vector sum of the forces. The direction of the change in momentum is that of the line of action of the resultant force.
- *Third Law*: Forces, arising from the interaction of particles, act in such a way that the force exerted by one particle on the second is equal and opposite to the force exerted by the second on the first and both are directed along the line joining the two particles. (Or, as usually expressed, *action* and *reaction* are equal and opposite.)

In the first law Newton implies what we call *frames of reference*. He has implied that a body at rest with respect to an observer can be analyzed in the same way as one moving past the observer at a constant velocity. That is, the same force applied to either body will change its state of motion in the same way.



Galileo Galilei (1564–1642).

The third law of action and reaction implies that there can be no single force in isolation. A force must act on a body and, when it does, the body acts on the source of the force. Consider a bat striking a ball. The bat is a mass in motion with a certain velocity. If it strikes the ball, it exerts a force on it and changes the ball's state of motion. At the instant of striking, the ball exerts a force on the bat in the opposite direction, thereby changing the bat's state of motion by slowing it down. If the bat misses the ball, it swings through the air with no appreciable change in motion because it has not exerted force on any body of substantial mass and therefore nothing has changed its state of motion. Strictly speaking, the bat is striking air molecules during its motion, thereby changing their state of motion and slightly reducing its speed. There is another force acting to change the direction of the bat's motion that causes it to move in an arc. If it were not for the force of the batter's hands the bat's motion would continue in a straight line. That is, if the batter lets go of the bat, it will fly off in a straight line; we will consider this phenomenon later.

René Descartes (1596–1650).

The first and third laws set the stage and the conditions for the second law. To appreciate Newton's approach, let us briefly look at it in historical perspective. It is common knowledge that Galileo (1564–1642) was tried for disobeying a directive from Catholic Church authorities not to state or publish his views of motion of the solar system. The next great natural philosopher to take a keen interest in the laws of mechanics was the Frenchman René Descartes (or in Latin, Renatus Cartesius) (1596–1650). Our cartesian coordinate system bears his Latin name. He decided from his studies that the most important property of a body in a mechanical system was what he called its *momentum*, the mass times the velocity, mv. We will deal with this in Chapter 6. Being a careful observer, he could not help but notice what had happened to Galileo, so he declined to publish his thoughts on mechanical systems. These were embodied in his manuscript *Le Monde*, which was first published in Amsterdam in 1662, 12 year after his death. The Reformation had by then reached Holland, so the fear of recrimination by ecclesiastical authorities was of little concern.

Meanwhile, back in England, the Great Plague was raging and everyone who had a relative in the country escaped from the city of London. Among these was Newton, who, according to legend, was contemplating the motion of a falling apple. A copy of *Le Monde* was given to him, and he was able to make the creative step shortly thereafter, although his approach was somewhat indirect. The concept of momentum, $m\mathrm{v}$, involves constant or zero velocity, which is embodied in his first law. If a force is applied against a body, the body resists with what he called an inertial force, namely, resistance to having its state of motion (momentum) changed. This is the principle embodied in the third law. If, however, a force is applied to a body for a given length of time, Δt, the momentum will be changed by $\Delta m\mathrm{v}$. He called the product of force and time an *impulse*, and he wrote the basic principle of the second law that the application of an impulse to a body caused a change in its momentum or

$$\mathrm{F}\Delta t = \Delta m\mathrm{v} \tag{4.1}$$

He recognized, however, that direction was equally important. That is, if an impulse was applied in the x direction, the momentum of the body would be changed in only the x direction. Thus Newton introduced the requirement of vectors in calculations.

If we divide Eq. 4.1 by Δt and consider the mass of the body to be constant, we may write

$$\mathrm{F} = m\frac{\Delta \mathrm{v}}{\Delta t} \tag{4.2}$$

where we must recognize that in most physically realizable situations F is not a constant but rather an average force; for example, the force of a bat against a baseball. In Chapter 3 we defined a Δ as a measurable quantity. Newton realized that he wanted to have a form of Eq. 4.2 for very small, or instantaneous, values of time. Because only tentative beginnings of calculus existed at that time, he proceeded to improve the methods. (G. Leibniz, a contemporary German mathematician, also refined the calculus, independently.) Combining the result of Eq. 3.5, $\mathrm{a} = d\mathrm{v}/dt$, with Eq. 4.2, we may write

$$\mathrm{F} = m\mathrm{a} \quad \mathrm{F} = m\mathrm{a} \tag{4.3}$$

To use Newton's great second law properly, we must include formally the two additional concepts: (1) This is a vector equation, and (2) the force, which is now instantaneous so the average is not required, is actually the net, or unbalanced, force in a given direction. We write this as the algebraic sum in each direction.

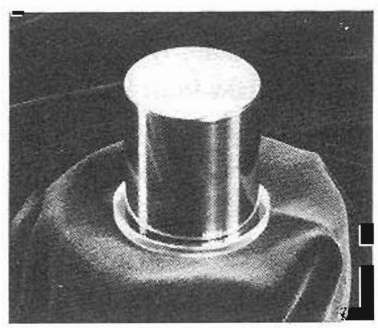
The forms of Newton's law that we will use are therefore

$$\sum F_x = ma_x, \ \sum F_y = ma_y, \ \sum F_z = ma_z \tag{4.4}$$

The concept of summation of forces can be understood from elementary examples. Consider a tug-of-war with equal numbers of people of equal strength pulling on a rope in opposite directions. If we consider a point in the center of the rope as the newtonian body, we conclude that the sum of forces acting on that body is zero and we will observe no acceleration. If a small child joins one of the sides and pulls, then the sum of forces is no longer zero, but instead there is a net force in the direction that the child pulls with magnitude equal to the force the child exerts. Hence, the rope will be accelerated in his direction.

It is convenient at this point to introduce the word *tension*, which is used to convey the transmission of a force through a rope. In this example, none of the contestant's hands are on the newtonian body at the center of the rope; yet, if the center is not being accelerated in either direction, the sum of forces at that point must be zero. If we insert a spring scale in the rope on either side of the center, we note that the same force is present, an example of *action* and *reaction*. Why does the spring scale not read zero, since we have just said that the sum of the force is zero? The answer lies with the third law of action and reaction. If you pull on one end of a spring, the other attached end is being pulled equally and in the opposite direction. Thus, although the sum of the forces is zero and the scale does not accelerate, the spring is nevertheless stretched, which causes the scale to read the value of whatever force is being applied to either end. We may perform the same measurement at any other point along the rope and the scale will read the same. The measured force in the rope at any point is called the *tension* in the rope. We could lower a curtain at the middle of the rope and tie one end of the rope to a wall and send that team home. The other team would not be aware of it, and a measurement in the rope would indicate the same tension. Suppose we went behind the curtain and cut the rope: How would Newton's law apply?

4.3 MASS



The standard kilogram, a platinum-iridium cylinder kept at the International Bureau of Weights and Measures in Sèvre, France.

Let us now reconsider our dilemma of defining mass and force. At this point we know from Chapter 3 how to measure acceleration, and we have stated that everyone has agreed to accept a certain block of material as having a mass of 1 kg. We do not yet know how to measure force, but we can devise a system to reproduce a given force, such as a pull on a rope with a spring scale to measure the same tension. If we exert this force on the standard kilogram, m_0, with no other forces such as friction or gravity to interfere, we can measure an acceleration $\mathbf{a}_0$. If we apply this same force to a different mass m_1, we measure a different acceleration $\mathbf{a}_1$. From Eq. 4.3 we may write for each experiment

$$F = m_0 a_0$$

$$F = m_1 a_1$$

and, equating the two because the forces are equal, we have

$$\frac{m_1}{m_0} = \frac{a_0}{a_1} \tag{4.5}$$

a relation independent of the value of the force. We thus have a method of measuring the mass of any other body in relation to a standard mass.

The unit of force can be defined in terms of mass, length, and time using Eq. 4.3.

$$\begin{aligned} F &= ma \\ &= [M]\left[\frac{L}{T^2}\right] \end{aligned}$$

where brackets contain the dimensionality of the quantities involved and M, L, and T stand for dimensions of mass, length, and time, respectively. Because the equation must balance dimensionally, force has units of mass × length/time^2 or kilogram-meter per second^2

$$F\left(\frac{ML}{T^2}\right)$$

In the SI system of units, this combination of units is called newton (N) for simplicity. *A force of 1 N is that force which causes a mass of 1 kg to be accelerated at a rate of 1 m/sec^2* (or 2 kg accelerated at 0.5 m/sec^2, and so on).

4.4 WEIGHT

A simple way to determine mass is to weigh it on a balance scale. In this method, a balance consists of a rod pivoted in the center so that the weighing pan on each side is equidistant from the center. (In Chapter 8 we will see how a balance scale may be constructed with arms of unequal length.) The unknown mass is placed on one side, and multiples or fractions of a standard kilogram are placed on the other side until a balance is achieved. In this way the magnitude of the unknown mass can be determined because both the unknown and known masses are being acted on by the same force, that of gravity.

The force of gravity can be expressed in terms of Eq. 4.3 by measuring g (acceleration due to gravitiy), the acceleration resulting from gravity. This can be done by noting that the rate of free fall of all objects in a vacuum (to eliminate air resistance) at a given point on earth is the same. The downward acceleration at sea level is approximately the same at all locations, or $g = 9.8\,\text{m/sec}^2$. So the force on an object of mass m resulting from gravity is, from Newton's second law

$$F = mg$$

and in the English language we call this force the *weight* of an object or

$$\text{Weight} = mg \tag{4.6}$$

Thus, for 1 kg

$$\text{Weight} = 1(\text{kg})g(\text{m/sec}^2) = g(\text{kg m/sec}^2)$$

$$= g \text{ newtons}$$

and 1 kg weighs 9.8 N. On the surface of the moon the acceleration of gravity is about one-sixth that of earth, so the weight of 1 kg will be one-sixth its weight on earth. In outer space at great distance from all other objects, the gravitational force will be near zero, and the kilogram will have almost zero weight. But its mass is unchanged. It takes the same force to produce a given acceleration in space as it does on the moon or on the earth.

In the English system we use units of *pounds* to express the weight of an object. Therefore, the pound is a force. Acceleration is ft/sec^2 and mass has units of

$$\frac{\text{Weight(pound)}}{g(\text{ft/sec}^2)} = m\left(\frac{\text{pound-sec}^2}{\text{ft}}\right)$$

The unit of mass in the English system is called the slug (from sluggish). At the surface of the earth the acceleration of gravity is 32.2 ft/sec^2 and 1 lb weight has a mass of 1/32.2 slugs.

4.5 APPLICATIONS OF NEWTON'S LAWS

4.5a. Zero Acceleration

It is seen from Eq. 4.4 that when $a = 0$ in a given direction the sum of forces, or net force, in that direction is zero. This fact can be used to gain information about the forces acting on an object when the object is not accelerated. When several forces act on an object but their effects cancel so that the object is not accelerated, the object is said to be in *equilibrium*.

EXAMPLE 4-1

A block rests on a table. What are the forces acting on the block? See Fig. 4-1.

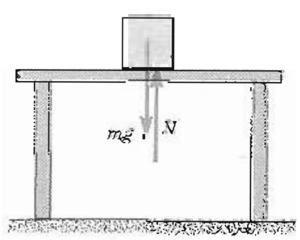


FIGURE 4-1 Example 4-1.

Solution Take the upward direction ($+y$ direction) as positive. We know that there is a force downward equal to the weight of the block. But because the block is not accelerating there must be an equal force upward, which we will call N for *normal* force, so that the sum of forces in the y direction is zero. Note here that the word *normal* is used in the mathematical sense of the direction perpendicular to a plane. We write Newton's law as

$$\sum F_y = 0$$

$$-mg + N = 0$$

$$N = mg$$

So the table exerts a force equal and opposite to the weight of the block; the table exerts a force on the floor equal to the sum of its weight and that of the block, the floor exerts an equal and opposite force on the table legs, and so forth.

EXAMPLE 4-2

A child pulls a toy boat through the water at constant velocity by a string parallel to the surface of the water on which he exerts a force of 1 N. What is the force of resistance of the water to the motion of the boat? See Fig. 4-2.

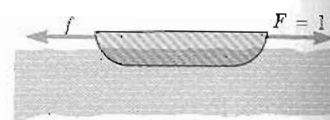


FIGURE 4-2 Example 4-2.

Solution Let F be the force parallel to the water of the string and f be the force of resistance of the water. Let us take the direction of F as the positive x direction. Because constant velocity means zero acceleration,

$$\sum F_x = 0$$

$$F - f = 0$$

$$f = F = 1\ \text{N}$$

EXAMPLE 4-3

Two ropes attached to a ceiling at the angles shown in Fig. 4-3 support a block of weight 50 N. What are the tensions T_1 and T_2 in the ropes?

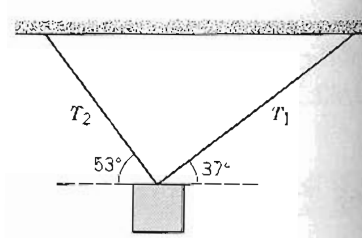


FIGURE 4-3 Example 4-3.

Solution Note here that the ropes exert forces both on the block and on the ceiling. The newtonian body of our concern is one through which all of the forces pass, namely the block. We therefore use the tensions acting on the block. We first draw a vector diagram of the forces (tensions) as in Fig. 4-4. If we examine the newtonian body, we see that it is not accelerating in either the x or y directions. We may therefore write

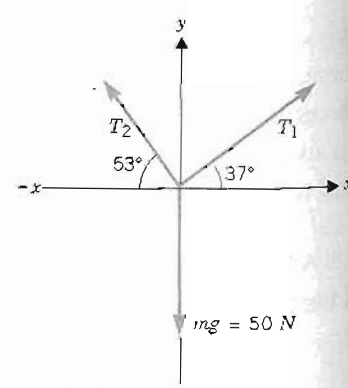


FIGURE 4-4 Example 4-3.

$$\sum F_x = 0, \quad \sum F_y = 0$$

By the component method of Chapter 2, we find the x and y components of the forces and substitute them into the equations.

$$\sum F_x = 0$$

$$T_1 \cos 37^\circ - T_2 \cos 53^\circ = 0$$

$$0.8T_1 - 0.6T_2 = 0$$

$$\sum F_y = 0$$

$$T_1 \sin 37^\circ + T_2 \sin 53^\circ - 50\,\text{N} = 0$$

$$0.6T_1 + 0.8T_2 - 50\,\text{N} = 0$$

We solve these simultaneously by substituting either T_1 or T_2 from one equation into the other. For example, from the $\sum F_x = 0$ equation we get

$$T_1 = \frac{0.6T_2}{0.8} = \frac{3}{4}T_2$$

Substituting into the second equation

$$0.6\left(\frac{3}{4}T_2\right) + 0.8\,T_2 - 50\,\text{N} = 0$$

$$1.25\,T_2 = 50\,\text{N}$$

$$T_2 = 40\,\text{N} \quad \text{and} \quad T_1 = \frac{3}{4}T_2 = 30\,\text{N}$$

4.5b. Constant Acceleration

In a constant acceleration situation we must examine the motion of the newtonian body in all of the cartesian directions. It may be accelerating in some directions but not in others. The direction or directions in which it is not accelerating may give additional information about the forces acting on the body.

EXAMPLE 4-4

A child pulls on a string attached to a 1-kg toy boat at an angle of 45° with a constant force of 2 N (Fig. 4-5). The boat goes from rest to a velocity of 0.2 m/sec in 0.5 sec. Assuming constant acceleration, what is the force of resistance of the water?

Solution

$$\sum F_x = ma_x$$

From Fig. 4-5, this becomes

$$2\,\text{N} \cos 45^\circ - f = ma_x$$

$$(2\,\text{N})(0.71) - f = (1\,\text{kg})a_x$$

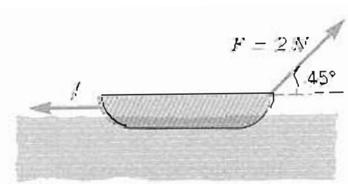


FIGURE 4-5 Example 4-4.

where we used the component of force in the direction of motion and f is the force of resistance of the water. We find a_x by the method of Chapter 3.

$$v_{0x} = 0, \quad v_{fx} = 0.2\,\text{m/sec}, \quad t_f = 0.5\,\text{sec}, \quad a_x = ?$$

$$v_{fx} = v_{0x} + a_x t_f$$

$$a_x = \frac{v_{fx} - v_{0x}}{t_f} = \frac{0.2\ \text{m/sec} - 0\ \text{m/sec}}{0.5\ \text{sec}} = 0.40\ \text{m/sec}^2$$

Substituting this result for a_x gives

$$1.42\ \text{N} - f = 0.40\ \text{N}$$

$$f = 1.02\ \text{N}$$

EXAMPLE 4-5

A block of mass 8 kg is released from rest on a frictionless incline that is at an angle of 37° with the horizontal (Fig. 4-6a). What is its acceleration down the incline?

Solution In this situation it is convenient to tilt our graph paper so that the x axis is along the incline, for that is the direction in which the acceleration is to be determined. The y axis will be perpendicular to the incline. The vector force diagram is shown in Fig. 4-6*b*. The only forces exerted on the block are mg downward and the normal force N on the block exerted by the plane that, as we indicated in Example 4-1, is perpendicular to the surface. The component of mg along the x axis, F_x, is determined by dropping a perpendicular from the end of the mg-force vector to the x axis. The angle between mg and this perpendicular is $\theta = 37°$ by the geometric rule that two angles are equal if their sides are mutually perpendicular: A is perpendicular to D, and B is perpendicular to C. Therefore,

$$\sin 37° = \frac{F_x}{mg}$$

$$F_x = mg \sin 37°$$

From Newton's second law, Eq. 4.4,

$$F_x = ma_x$$

$$a_x = \frac{F_x}{m}$$

$$= \frac{mg \sin 37°}{m}$$

$$= g \sin 37° = 9.8\ \text{m/sec}^2 \times 0.6$$

$$= 5.9\ \text{m/sec}^2$$

Two important points can be seen in this simple problem:

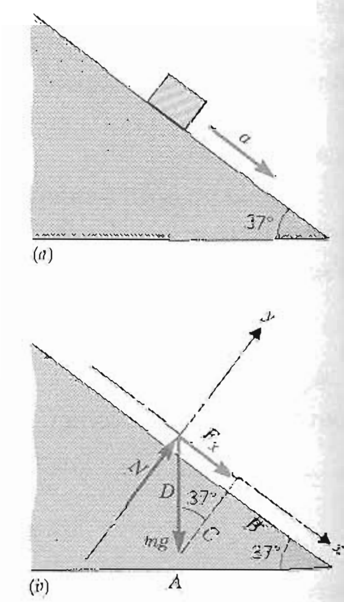


FIGURE 4-6 Example 4-5. (*a*) Diagram of the problem. (*b*) Force diagram.

1 Because the acceleration is independent of the mass, all masses starting from rest at the same height on the same plane will have the same acceleration and, therefore, reach the bottom at the same time.
2 The acceleration is less than the acceleration of gravity because only a component of the force of gravity on the body is directed down the plane.

EXAMPLE 4-6

Masses of 2 kg and 4 kg connected by a cord are suspended over a frictionless pulley (Fig. 4-7*a*). What is their acceleration when released?

Solution Before solving this problem, we note three important facts. First, because the pulley is frictionless, the tension in the rope is the same on both sides. If it were not, the cord would slide over the pulley until the tensions were the same. Second, the tensions are not the same as in a static situation; that is, we *cannot* equate $T = mg$ because $\sum F_y \neq 0$. Third, there are two newtonian bodies and we must write an equation for each, but note that while m_1 moves upward with a positive acceleration, m_2 moves with an acceleration having the same magnitude but directed downward. The force diagrams for the two bodies are given in Fig. 4-7*b*.

For body m_1 we write

$$\sum F_y = m_1 a$$

$$T - m_1 g = m_1 a$$

$$T = m_1(g + a)$$

For body m_2

$$\sum F_y = m_2 a$$

and, noting that because upward motion was chosen as positive for body 1, the downward acceleration of body 2 must be negative, we write

$$T - m_2 g = m_2(-a)$$

$$T = m_2(g - a)$$

Substituting the T from the body 1 equation, into the equation for body 2, as the tensions are the same, we obtain

$$m_1(g + a) = m_2(g - a)$$

Rearranging terms,

$$a(m_1 + m_2) = g(m_2 - m_1)$$

$$a = g\frac{m_2 - m_1}{m_1 + m_2}$$

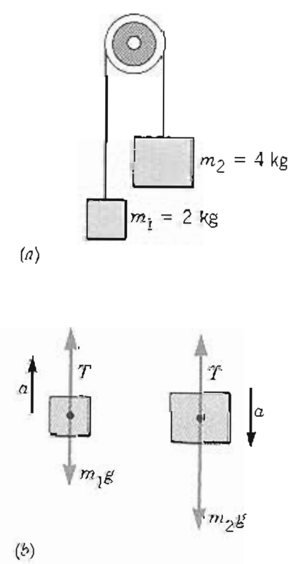


FIGURE 4-7 Example 4-6 (*a*) Diagram of the problem. (*b*) Force diagram for each of the masses.

$$= 9.8\ \text{m/sec}^2 \times \frac{4\ \text{kg} - 2\ \text{kg}}{2\ \text{kg} + 4\ \text{kg}} = 3.3\ \text{m/sec}^2 \quad \text{for body 1}$$

and

$$a = -3.3\ \text{m/sec}^2 \quad \text{for body 2}$$

4.6 FRICTION

We have to exert a steady force to drag an object at constant velocity across the floor. Because the velocity is constant, the acceleration is zero and the sum of forces on the object is correspondingly zero. This means that there is a force equal and opposite to the force that we exert that resists the motion of the object. However, our tug-of-war example does not apply here, for if we stop pulling the object the resistive force does not start to pull it in the opposite direction. Nearly all surfaces have a certain amount of roughness, visible under a microscope, and it is the breakage of these rough protrusions or the rising over them that causes the resistance to motion. This resistive force is called the *force of friction*. The earlier example, 4-2, of the boy with the boat is another type of friction, that of water resisting the motion of an object moving through it. But why consider friction at all in a book about semiconductors and their circuits? Because we commonly experience friction of the types we are discussing here and thus they are easier to comprehend. We will later consider the motion of electrons through a solid under the influence of electric forces. The motion of the electrons is impeded by their banging into the atoms in the solid and losing energy in the process. This is another type of friction.

Returning now to the behavior of an object with an opposing frictional force, we must leave First Principles temporarily and rely on experimental data. There are two types of friction, *static* and *kinetic*. The starting friction is called *static*. The friction of motion is called *kinetic*. We observe from experience that it is harder to start an object moving across a floor than it is to maintain its motion; static friction is larger than kinetic friction. We will only consider kinetic friction. If we wish to measure the force of kinetic friction, we have only to measure the force required to keep an object in motion at constant velocity on a level surface. If we add a weight equal to that of the object on top of it, we find it takes twice the force to keep it moving at constant velocity; with the object weighing three times as much, then three times the force is required. We would correctly conclude that the force of friction is proportional to the weight of the object. But, as can be seen from Fig. 4-8, it is equivalent to say that the force of friction is proportional to the normal force because $mg = N$. Either way we say that the force of friction is proportional to the force pushing the surfaces together. Therefore

$$f \propto N \tag{4.7}$$

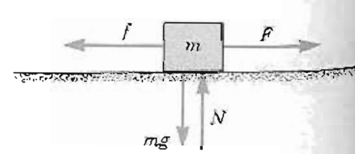


FIGURE 4-8 Forces on a mass that give rise to a force of friction f.

Now we know that it is easier to pull or push an object across ice than across a floor. Therefore, we may transform the proportionality of Eq. 4.7 to an equality by introducing a constant that characterizes the surface. We customarily use the Greek letter μ (mu) for this, and μ is called the *coefficient of friction*. Thus

$$f = \mu N \tag{4.8}$$

EXAMPLE 4-7

A force of 10 N is required to keep a box of mass 20 kg moving at a constant velocity across a level floor (Fig. 4-9). What is the coefficient of friction?

Solution Because the velocity is constant, $a_x = 0$ and $a_y = 0$, and

$$\sum F_x = 0$$

$$F - f = 0$$

$$f = 10\,\text{N}$$

and

$$\sum F_y = 0$$

$$N - mg = 0$$

$$N = mg$$

But

$$f = \mu N$$

$$f = \mu mg$$

or

$$\mu = \frac{f}{mg}$$

$$= \frac{10\,\text{N}}{20\,\text{kg} \times 9.8\,\text{m/sec}^2}$$

$$\mu = 0.05$$

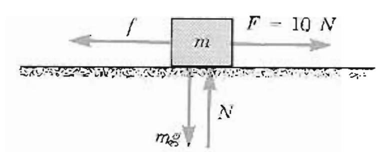


FIGURE 4-9 Example 4-7.

Suppose that the surface is not level but is inclined by an angle θ with respect to the horizontal, as in Fig. 4-10. We see that the component of the weight mg along the axis perpendicular to the plane is $-mg\cos\theta$ and, by Newton's second law, because $a_y = 0$, the normal force N exerted by the plane on the block is $N = mg\cos\theta$. Thus, Eq. 4.8 can be written in a more general form as

$$f = \mu mg \cos\theta \tag{4.9}$$

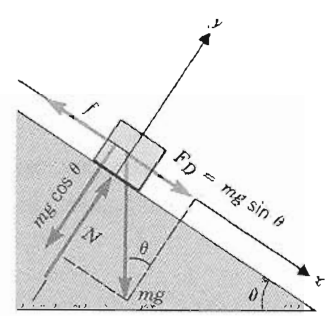


FIGURE 4-10 A block in an inclined plane. The normal force is equal to the component of the weight perpendicular to the plane $mg\cos\theta$ and the force of friction is equal to $-\mu mg\cos\theta$

when $\theta = 0, f = \mu mg$ and is at a maximum. When $\theta = 90°$, the incline is standing vertically and there is no force pushing the surfaces together and $f = 0$.

EXAMPLE 4-8

A block is placed on a plane inclined to the horizontal at 37°. The coefficient of friction between the plane and the block is $\mu = 0.4$. When the block is released, what is its acceleration down the plane?

Solution The forces along the plane are the force of friction f upward and the component of the force of gravity F_D downward (see Fig. 4-10). Choose the downward direction as positive and write Newton's second law.

$$\sum F_{\text{plane}} = ma_{\text{plane}}$$

$$F_D - f = ma_{\text{plane}}$$

We have seen in Example 4-5 that

$$F_D = mg \sin\theta$$

and Eq. 4.9 gives the expression for f

$$mg \sin\theta - \mu mg \cos\theta = ma_{\text{plane}}$$

Solving for the acceleration, we obtain

$$a_{\text{plane}} = \frac{mg \sin\theta - \mu mg \cos\theta}{m}$$

$$= g \sin\theta - \mu g \cos\theta$$

Substituting the known quantities

$$a_{\text{plane}} = 9.8\,\text{m/sec}^2 \times 0.6 - 0.4 \times 9.8\,\text{m/sec}^2 \times 0.8$$

$$= 2.74\,\text{m/sec}^2$$

We see that the mass cancels; that is, all blocks with the same coefficient of friction will have the same acceleration down the plane.

PROBLEMS

4.1 A 50-N weight is suspended by a rope from the ceiling. A horizontal force pulls it sideways, causing the rope to make an angle with the ceiling of 53°. When the weight is in equilibrium, what is the force?

4.2 A 40-N weight is suspended by a rope from the ceiling. Another rope pulls horizontally on it sideways so that the suspending rope makes an angle of 60° with the ceiling. What are the tensions in the ropes?

4.3 A 50-N weight is suspended by a rope from the ceiling. A horizontal force of 40 N pulls on the weight in the x direction. (a) What is the angle that the rope makes with the ceiling? (b) What is the tension on the rope?

4.4 A 100-N weight is suspended by ropes as shown in Fig. 4-11. Find the tension on each rope.

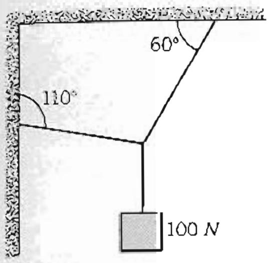


FIGURE 4-11 Problem 4.4.

4.5 A force of 50 N acting at 37° above the horizontal pulls a block along the floor with constant velocity. If the coefficient of friction between the block and the floor is 0.2, what is the mass of the block?

Answer: 23.4 kg.

4.6 A 500-kg box is to be lowered down a ramp at constant velocity. The ramp makes an angle of 30° with the ground. The coefficient of friction between the ramp and the box is 0.7. (a) What force applied parallel to the ramp is needed? Must the box be pushed down or held back? (b) Repeat (a) if the coefficient of friction is 0.2.

4.7 A constant horizontal force of 50 N acts on a body that is resting on a smooth, frictionless horizontal plane. The body is observed to go from rest to $v = 5$ m/sec in 10 sec. What is the mass of the body?

Answer: 100 kg.

4.8 A body of mass 5 kg rests on a horizontal frictionless plane. A force of 10 N is applied at an angle of 37° above the plane for 5 sec. How far has the body moved in that time?

4.9 An electron of mass 9×10^{-31} kg leaves the heated filament of a vacuum tube with $v_0 = 0$ m/sec and travels in a straight line toward a plate 1 cm away. It arrives there with a velocity of 7×10^6 m/sec. Find the magnitude of the acceleration and the accelerating force.

4.10 A 5-kg mass is attached to the end of a string with a breaking strength of 100 N. What is the maximum acceleration that the mass can be given by pulling the string in the upward direction?

Answer: 10.2 m/sec^2.

4.11 Three blocks of mass—$m_1 = 3$ kg, $m_2 = 4$ kg, $m_3 = 6$ kg—resting on a frictionless table and connected by strings with tensions T_1 and T_2 are being pulled to the right by a force of 6 N (Fig. 4-12). (a) What is the acceleration of the blocks? (b) What are the tensions in the strings?

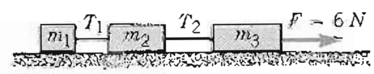


FIGURE 4-12 Problem 4.11.

4.12 What horizontal force is required to drag a 5-kg block along a horizontal surface, with a coefficient of friction of 0.5, at a constant acceleration of 1 m/sec^2?

4.13 A block of 8 kg is held on an incline at 37°. The coefficient of friction between the block and the incline is $\mu = 0.1$. When the block is released, what will be its acceleration?

4.14 Consider an 8-kg block on a frictionless plane inclined at 37° to the horizontal, as in Fig. 4-6a. Suppose a force of 40 N is applied to the block upward along the plane. (a) What will be the acceleration of the block? (b) If the upward force applied is 60 N, what will be its acceleration? (c) What force is required along the plane to hold the block motionless?

Answer: (a) 0.90 m/sec^2 downward, (b) 1.60 m/sec^2 upward, (c) 47.18 N.

4.15 (a) What constant force acting parallel to a 37° plane is required to push a 10-kg block up the plane at constant speed if the coefficient of friction is 0.5? (b) What force is required to push it up with an acceleration of 2 m/sec^2?

4.16 Block A rests on a frictionless plane and the connecting cord passes over a frictionless pulley with block B attached to it (Fig. 4-13). What is the acceleration of block A along the plane when it is released?

Answer: 0.67 m/sec^2 down the plane.

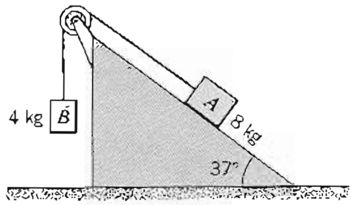


FIGURE 4-13 Problem 4.16.

4.17 An object is hung by a rope to the ceiling of an elevator. When the elevator rises at constant speed, the tension in the rope is 50 N. (a) What is the tension when the elevator is accelerating upward at 3 m/sec^2? (b) What is the acceleration of the elevator if the tension is 30 N?

4.18 An object slides down a 37° incline with constant velocity. After reaching the bottom, it is launched up the incline with an initial velocity of 5 m/sec. How far up the incline will it move before it stops?

Answer: 1.06 m.

4.19 In Fig. 4-14, the masses of blocks A, B, and C are 5 kg, 20 kg, and 10 kg, respectively. The blocks are observed to move with constant velocity. What will be the acceleration of blocks A and B when block C is removed?

Answer: 0.65 m/sec^2.

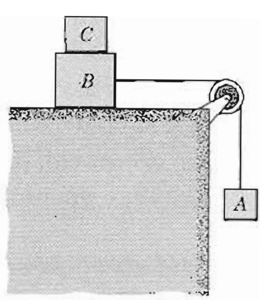


FIGURE 4-14 Problem 4.19.

4.20 A 40-N block is connected to a second block by a light rope passing over a frictionless pulley, as in Fig. 4-15. The coefficient of friction between the blocks and the inclines is 0.25. If the 40-N block moves up the plane at constant velocity: (a) What is the weight of the second block? (b) What is the tension in the rope? (c) Suppose that the 40-N block is replaced by a 100-N block and the coefficient of friction remains unchanged, what will be the acceleration of the blocks?

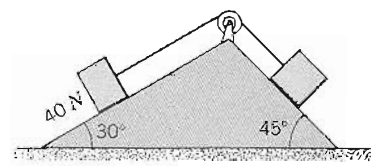


FIGURE 4-15 Problem 4.20

4.21 A rope of breaking strength 800 N is to be used to drag a box at constant velocity on a horizontal surface. The rope pulls the box at some angle θ above the horizontal. If the coefficient of friction is 0.3, what is the maximum weight of the box that can be moved without the rope breaking?

Answer: 2784 N.

4.22 A 10-kg ball is hung by a rope from the ceiling of a car. The maximum tension that the rope can withstand is 500 N. (a) What is the maximum horizontal acceleration that the car can reach without the rope breaking? (b) What is the angle between the rope and the vertical for that acceleration?

Answer: (a) 49.03 m/sec^2, (b) 78.7°.

4.23 A block is held against the front vertical wall of a railroad car, as in Fig. 4-16. The coefficient of friction between the block and the wall is 0.4. When the train begins to accelerate, the block is released and begins to slide down the wall with an acceleration of 9.0 m/sec^2. What is the horizontal acceleration of the train?

Answer: 2 m/sec^2.

FIGURE 4-16 Problem 4.23.

4.24 In Fig. 4-17, block B of mass 5 kg is being pulled by a force $F = 100$ N. The mass of block A is 2 kg and the coefficient of friction between all surfaces is 0.2. The pulley is frictionless. Find the acceleration of the blocks and the tension in the rope.

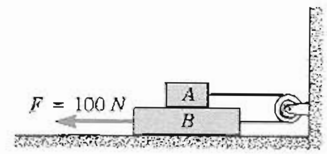


FIGURE 4-17 Problem 4.24.

5.1 INTRODUCTION

The terms work, energy, and power are common words in English. In physics we require precise definitions so that these terms can be formulated mathematically. Readers will find that definitions in physics do not always match the usage of the words. For example, although a student may do homework, by the physics definition of work none is being done, although at the biological cellular level, chemical work is being done. We will not consider chemical work in this book. It is important that we consider *mechanical* work, energy, and power, for it is the treatment of these terms from First Principles that will be applied directly to electrical circuits. It is therefore essential that the physics definitions of these terms be learned carefully.

5.2 WORK

In this first treatment of work we will restrict our consideration to that done by a constant force. Our definition of an amount of work ΔW done by a constant force F acting on a body is: The product of the distance the body is moved in a given direction by the component of the force in that direction,

$$\Delta W = F_s \, \Delta s \tag{5.1}$$

where F_s represents the component of force in the direction Δs. If F_s is 1 N and Δs is 1 m, then the work ΔW done on the body by the force F is 1 newton-meter (N-m). We define a new unit, 1 N-m = 1 joule (pronounced in America as jewel) with symbol J.

EXAMPLE 5-1

A box is pushed 3 m at constant velocity across a floor by a force F of 5 N parallel to the floor. (a) How much work was done on the box by the force F, which clearly opposes friction (see Fig. 5-1). (b) How much work is done on the box by the force of friction?

Solution

(a) $W = 5\,\text{N} \times 3\,\text{m} = 15\,\text{J}$

(b) Because $a = 0$

$$\sum F_x = 0$$

$$F - f = 0$$

$$f = 5\,\text{N}$$

FIGURE 5-1 Example 5-1.

The component of **f** in the direction of the displacement vector $\Delta \mathbf{x}$ is $-f$ (because frictional forces always act against the motion); therefore

$$\Delta W = (-f)\,\Delta x$$
$$= -15\text{ J}$$

Although work may be positive or negative, it has no direction and is therefore a *scalar* quantity. We will now show that work meets the condition of the vector dot product of Chapter 2, which was defined as a scalar. Suppose the force pulling a box across a floor is not in the direction of motion but is in the direction shown in Fig. 5-2. Following the definition of work, we must take the component of the force in the direction of motion

$$\Delta W = F_x\,\Delta x$$
$$= F\cos\theta\,\Delta x$$
$$\Delta W = F\,\Delta x\cos\theta \qquad (5.2)$$

which, by Eq. 2.1, is

$$\Delta W = \mathbf{F}\cdot\Delta\mathbf{x} \qquad (5.3)$$

Because Eqs. 5.2 and 5.3 are expressions of the general relation of Eq. 2.1, the F and Δx in Eq. 5.2 are the magnitudes, and we thus have a scalar. Eq. 5.2 has an important conceptual implication. Suppose we carry a weight mg across the room. If it is initially placed in our hands and we carry it slowly, without appreciable acceleration, the only force we exert is in the upward direction.

$$\sum F_y = 0$$
$$-mg + F_y = 0$$
$$F_y = mg$$

If we move a distance Δx, from our definition of work

$$\Delta W = F_y\,\Delta x\cos\theta$$

the angle θ between the force direction and the motion direction is 90°. Therefore we do no work.

We may use the definition of Eq. 5.2 to treat formally the work of friction of Example 5-1.

$$\Delta W = f\,\Delta x\cos\theta$$

but $\theta = 180°$ and $\cos 180° = -1$, so we could have simply taken the magnitude of friction times the distance and the $\cos\theta$ would have yielded the correct sign.

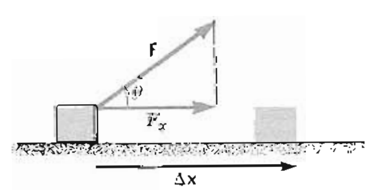


FIGURE 5-2 Force and motion are not in the same direction.

5.3 POTENTIAL ENERGY

Consider now that we lift a weight mg at constant velocity a distance y from the floor. Because gravity exerts a downward force mg, we must exert an upward force of equal magnitude over a distance y and therefore have done an amount of work mgy on the weight. If we lower it back to the floor, we still exert an upward force mg, but now the motion is in a direction opposite to the force, or $\theta = 180°$, and the work done by us is $-mgy$ and therefore no net work has been done in the round trip. Suppose we wish to move a weight mg from the floor to a shelf a distance x across the room. There are many paths that we may take, some of which are shown in Fig. 5-3. An examination of these paths shows that each move in the y direction contributes positive or negative work amounts whose sum must be mgy, whereas motion in the x direction contributes no work. We are thus able to draw a very important conclusion. *Work done against the gravitational force is independent of the choice of path between any two fixed endpoints.*

Suppose an object is placed at a height y in a gravitational field, as in Fig. 5-4. If it descended from y, the gravitational force mg on the object would be capable of doing work equal to the force times the displacement mgy. Therefore, because the gravitational force is potentially able to do work on the object, we say that it has a *potential energy* E_p equal to mgy. (Note that the unit of potential energy is the same as that of work, that is, the joule)

$$E_p = mgy \tag{5.4}$$

That is, if we have lifted it by doing work mgy on it, then it has gained a potential energy of mgy, where we use the positive value because the work was put into it. Note that there is a direct relation between the work done on an object in a gravitational field and its gain in potential energy. The measure of y must be considered with care. Suppose an object is lifted above a table to a height y_1. It has $E_p = mgy_1$ with respect to the table. But if the table is at a height y_3 above the floor, the object has $E_p = mg(y_1 + y_3)$ with respect to the floor (see Fig. 5-5).

Thus, for potential energy, a reference level must always be specified. If we lift an object two different distances above a table, then we may state the difference in potential energy between the two positions. If we included the distance above the floor for each, then when we take the difference in their E_p, the distance above the floor will cancel,

$$\Delta E_p(\text{with the table as the reference level})$$

$$= mgy_2 - mgy_1 = mg(y_2 - y_1)$$

$$\Delta E_p(\text{with the floor as the reference level})$$

$$= mg(y_2 + y_3) - mg(y_1 + y_3)$$

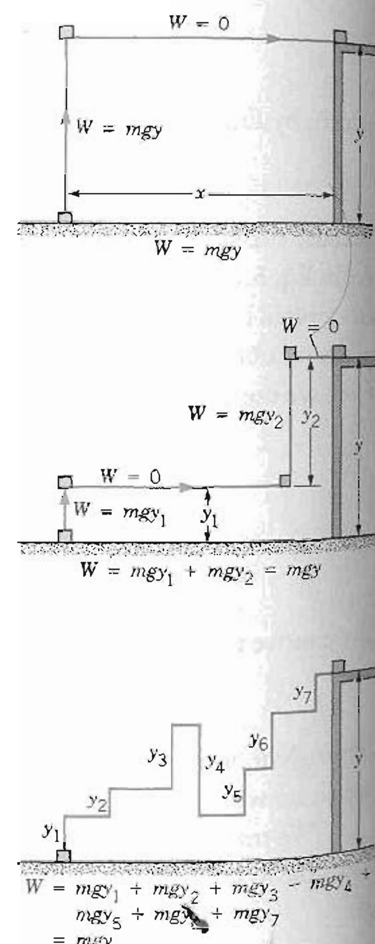


FIGURE 5-3 Examples of different paths used in raising a mass to a height y.

$$= mgy_2 - mgy_1 = mg(y_2 - y_1)$$

We see in this answer that only the difference in heights needs to be specified to give the relative difference in potential energy. We will use this concept in electricity to specify the relative difference in potential energy of a charged particle in two different positions in an electric field. To apply the concept of potential energy in later chapters we will need to expand our consideration of work to that done by a variable force.

5.4 WORK DONE BY A VARIABLE FORCE

We have seen that when the force acting through a distance Δx is constant, then the work done may be written as

$$\Delta W = \mathbf{F} \cdot \Delta \mathbf{x} = F_x \, \Delta x \tag{5.3}$$

Suppose that at each small displacement of the motion the force has a different value. Then we would write Eq. 5.3 as

$$W = F_{x1}\,\Delta x_1 + F_{x2}\,\Delta x_2 + F_{x3}\,\Delta x_3 + \cdots F_{xN}\,\Delta x_N$$

or

$$W = \sum_{i=1}^{N} F_{xi}\,\Delta x_i \tag{5.5}$$

A sketch of this summation is shown in Fig. 5-6 for the work done in moving a body from $x = a$ to $x = b$. We see that each vertical segment of Δx_i width has associated with it an average F_{xi}. The total work is the sum of the areas of these segments. If we make the width of each segment very small so that $\Delta x \to 0$, the number of segments required to obtain the area under the curve approaches infinity and the sum of these infinitesimal areas becomes the precise area under the curve of Fig. 5-6, or the total work. This is the definition of an integral. That is, Eq. 5.5 as $\Delta x \to 0$ becomes

$$W = \int_a^b F_x\,dx \tag{5.6}$$

Furthermore, by the definition of an integral, and with reference to Fig. 5-6, we see that work is the area under the F_x versus the x curve. In the derivation of Eq. 5.6 we have assumed that F, although not constant in magnitude, is always in the direction of the displacements Δx's. In the more general case where F and the general displacement Δs's are not in the same direction, the expression for the work becomes

$$dW = \mathbf{F} \cdot d\mathbf{s} \tag{5.7}$$

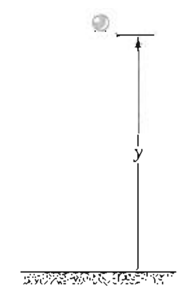


FIGURE 5-4

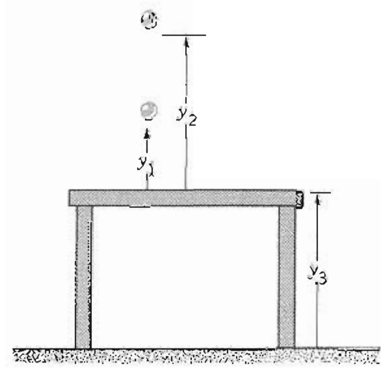


FIGURE 5-5

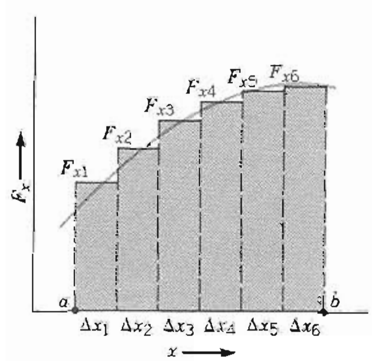


FIGURE 5-6 Area under the curve obtained by summation of the area of rectangles.

or the integral form

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{s} \tag{5.7'}$$

where the dot product takes care of changes in orientation between the vectors **F** and $d\mathbf{s}$.

5.5 KINETIC ENERGY

The word "kinetic" is used frequently in all branches of science and is from the Greek word for motion. If work is done on a body that changes its state of motion, namely, its velocity, we say that the work has caused the body to gain or lose *kinetic energy*, E_k. We will show two derivations of kinetic energy, the first for the restricted condition of a constant force and the second for a variable force. In both cases we will consider motion in the x direction alone.

If the force is constant and the initial position is $x = 0$, we may write the definition of work as

$$W = F_x x$$

which from Newton's second law (Eq. 4.4) can be written as

$$W = ma_x x$$

or, if we take a only in the x direction, we may drop the subscript and write

$$W = max \tag{5.8}$$

Because the force is constant and we are considering a single body of constant mass, the acceleration is constant and we may use Eq. 3.11

$$v^2 - v_0^2 = 2\,ax \tag{3.11}$$

where v_0 is the velocity at $x = 0$ and v is the velocity at x. Substituting this equation into Eq. 5.8 we obtain

$$W = m\left\{\frac{v^2 - v_0^2}{2}\right\}$$

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \tag{5.9}$$

Thus the work done on a body that changes its velocity actually changes the quantity $\frac{1}{2}mv^2$, which is called the *kinetic energy* E_k.

$$E_k = \frac{1}{2}mv^2 \tag{5.10}$$

Because change in kinetic energy is equal to work and work is a scalar quantity, kinetic energy is also a scalar quantity and the unit of kinetic energy is the same as that of work, that is, the joule.

We obtain the same result if the applied force is not constant but is variable. Let us consider the x axis motion again so that

$$W = \int_{x_0}^{x} F\,dx$$

In this case F must remain inside the integral, but we can substitute Newton's second law for it, $F = ma$

$$W = m\int_{x_0}^{x} a\,dx \tag{5.11}$$

And we may substitute for a from Eq. 3.13

$$a = v\frac{dv}{dx} \tag{3.13}$$

Thus Eq. 5.11 becomes

$$W = m\int_{x_0}^{x} v\frac{dv}{dx}dx$$

$$W = m\int_{v_0}^{v} v\,dv$$

where we have changed the limits of integration because the velocity is v_0 at x_0 and v at x. This integrates to

$$W = m\frac{v^2}{2}\Big]_{v_0}^{v}$$

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \tag{5.9}$$

which is the same result as found before for a constant force. Note that in both derivations we have used $F = ma$ where F is the net, or resultant, force. If $a = 0$, then $F = 0$ and there is no net, or resultant, force and hence there can be no change in the kinetic energy. Eq. 5.9 is known as the *work-energy theorem*, which states that *the work done by the resultant force acting on a particle is equal to the change in kinetic energy of the particle.*

5.6 ENERGY CONSERVATION

We define a *mechanically conservative* system as one in which *no* energy enters or leaves the system (as heat, radiation, or such). Therefore, the system's initial energy is unchanged, which is the same as saying it is conserved. This fact simplifies the solution of many types of problems because we do not have to calculate the acceleration. For conservative systems we know that the total energy in state 1 is the same as that in state 2. Because the total energy is the sum of the potential and kinetic energies, we write

$$(E_k + E_p)_{\text{initial}} = (E_k + E_p)_{\text{final}} \tag{5.12}$$

We can verify explicitly the conservation of the total energy with the simple example shown in Fig. 5-7.

Let us launch an object of mass m from a point y_1 above the floor with an initial velocity v_1. Owing to the gravitational force that acts on the object, its velocity decreases as it rises. Sometime later, the velocity of the object will be v_2 and its position y_2. We can relate this new velocity and position to the initial velocity and position by means of Eq. 3.11.

$$v_2^2 - v_1^2 = 2(-g)(y_2 - y_1)$$

Rearranging terms,

$$v_2^2 + 2gy_2 = v_1^2 + 2gy_1$$

If we divide both sides by 2 and multiply by the mass of the object m, we get

$$\frac{1}{2}mv_2^2 + mgy_2 = \frac{1}{2}mv_1^2 + mgy_1 \tag{5.13}$$

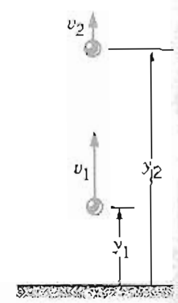
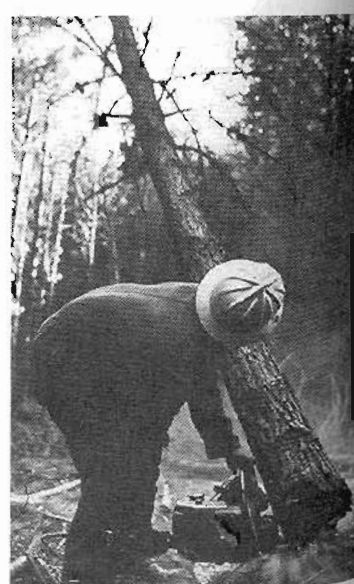


FIGURE 5-7

Thus, although the velocity of the object (and therefore its E_k) decreases as it rises, this decrease in E_k is compensated by an increase in E_p in such a way that the sum of E_k and E_p remains constant and equal to the sum of the initial kinetic and potential energies. That is, total energy is conserved.


A falling tree illustrates the conversion of potential energy into kinetic energy.

EXAMPLE 5-2

Suppose a ball is dropped from a height $h = 10\,\text{m}$. What is its velocity just before it strikes the ground?

Solution

$$E_{ki} + E_{pi} = E_{kf} + E_{pf}$$

$$0 + mgh = \frac{1}{2}mv^2 + 0$$

$$v = \pm\sqrt{2gh} = \pm\sqrt{2 \times 9.8\,\text{m/sec}^2 \times 100\,\text{m}} = -14\,\text{m/sec}$$

where the negative sign is chosen because the motion is downward.

The pendulum is another simple example of the conservation of energy. Let us assume an idealized pendulum that swings in a vacuum so that there is no energy lost to air friction and that there is no frictional loss at the pivot (see Fig. 5-8). If we start the pendulum by pulling it to one side and releasing it with no initial velocity, it has an initial potential energy of mgh_0, which is also the total energy. If there is no energy loss during its subsequent motion, it must always have this amount of total energy. When it is released, it begins to fall and potential energy is lost. But as it falls it picks up speed and thereby gains kinetic energy. At the bottom of its path, $h = 0$ and all its energy is kinetic. As it starts to rise again, the kinetic energy is converted to potential energy. Thus, if h_0 is its initial height, mgh_0 is the energy of the pendulum and the sum of the potential and kinetic energies at all other positions must equal this value. We may write this as

$$E_{p0} + E_{k0} = E_{p2} + E_{k2}$$

$$mgh_0 + 0 = mgh_2 + \frac{1}{2}mv_2^2$$

The string does no work on the pendulum because of the definition (Eq. 5.7)

$$dW = \mathbf{F} \cdot d\mathbf{s} = F\cos\theta\, ds$$

The angle θ is that between the string direction and ds, the instantaneous direction of motion. This angle is always 90°, so that dW due to the string is always zero.

If the system loses energy or energy is put into the system, it is no longer a mechanically conservative system. Later, when we understand other types of energy and can enlarge the system to include all sources of input and output, we will again develop the concept of conservation of energy. For now, however, let us say that all energy must be accounted for and use the term *accountability of energy*. We become accountants and keep books of assets and liabilities. All initial energy plus any energy put in, E_{in}, is on the left side of the ledger as an asset. All energy converted to another form or escaping from the system, E_{out}, may go on the right side. Thus, Eq. 5.12 is written as

$$E_{ki} + E_{pi} + E_{in} = E_{kf} + E_{pf} + E_{out} \qquad (5.14)$$

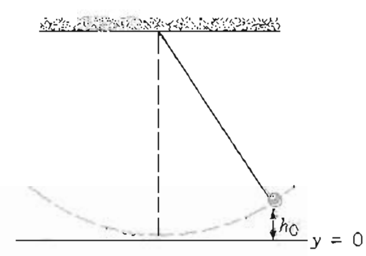


FIGURE 5-8

EXAMPLE 5-3

A skier is on a 37° slope of length $s = 100$ m (Fig. 5-9). The coefficient of friction between his skis and the snow is 0.2. If he starts from rest, what is his velocity at the bottom of the slope?

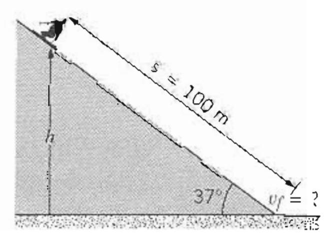


FIGURE 5-9 Example 5-3.

Solution

$$E_{ki} + E_{pi} + E_{in} = E_{kf} + E_{pf} + E_{out}$$

No energy is put in, but there is energy lost to work against friction, or $E_{out} = |W_{friction}| = |\mathbf{f} \cdot \mathbf{s}|$

As before, let us tilt our coordinate axis so that the slope becomes the x axis and the normal becomes the y axis.

$$\sum F_y = 0$$

$$N - mg\cos 37^\circ = 0$$

$$N = mg\cos 37^\circ$$

$$f = \mu N = \mu mg\cos 37^\circ$$

$$E_{\text{out}} = |\mathbf{f}\cdot\mathbf{s}| = \mu mg\cos 37^\circ(s)$$

$$0 + mgh + 0 = \frac{1}{2}mv_f^2 + 0 + \mu mg\cos 37^\circ(s)$$

The mass cancels, and we solve for v_f

$$v_f = [2(gh - \mu gs\cos 37^\circ)]^{1/2}$$

$$= [2(9.8\,\text{m/sec}^2 \times 100\,\text{m} \times \sin 37^\circ$$

$$- 0.2 \times 9.8\,\text{m/sec}^2 \times 100\,\text{m} \times \cos 37^\circ)]^{1/2}$$

$$= 29.4\,\text{m/sec}$$

5.7 POWER

Different persons or different machines may take different amounts of time to do the same amount of work. The term used to describe this rate of performance of work is *power*.

$$\text{Power} = \frac{\text{work done}}{\text{time taken}}$$

$$P = \frac{W}{t} \tag{5.15}$$

Work is measured in joules, time in seconds, therefore the unit of power is joules per second (J/sec). We introduce a new unit: 1 J/sec = 1 watt (W). Conversely, work (or energy) is equal to power × time,

$$W(\text{joules}) = P(\text{watts})t(\text{sec})$$

The symbol used for watt is W. A 100-W light bulb uses 100 J of electrical energy each second. Your electric light bill is in kilowatt-hours. A kilowatt-hour is the energy dissipated by a device that uses 10^3 W for a period of 1 h, that is, 1 kwh = 10^3 J/sec $\times 3600$ sec = 3.6×10^6 J.

We may develop another expression for power because work is defined by Eq. 5.3 as the dot product of force F and displacement Δs, where Δs can be in the x direction. From Eq. 5.15

$$P = \frac{\mathbf{F} \cdot \Delta x}{\Delta t}$$

or for infinitesimally small displacements $d\mathbf{x}$

$$P = \mathbf{F} \cdot \frac{d\mathbf{x}}{dt}$$

but

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \text{ the velocity}$$

and therefore an alternative equation for power is

$$P = \mathbf{F} \cdot \mathbf{v} \tag{5.16}$$

There is an engineering unit of power that we might introduce in passing: horsepower (hp). This is defined as

$$1\ \text{hp} = 746\ \text{W}$$

EXAMPLE 5-4

A tractor can exert a force of 3×10^4 N while moving at a constant speed of 5 m/sec. What is its horsepower?

Solution

$$\begin{aligned} P &= \mathbf{F} \cdot \mathbf{v} \\ &= 3 \times 10^4\ \text{N} \times 5\ \text{m/sec} \\ &= 1.5 \times 10^5\ \text{W} \left(\frac{1\ \text{hp}}{746\ \text{W}}\right) = 200\ \text{hp} \end{aligned}$$

PROBLEMS

5.1 A car is pulling a trailer with a force of 900 N on a level road. The car is moving at 70 km/h. How much work is done by the car on the trailer in 15 min?

5.2 A 50-kg box is being pushed on a horizontal surface, at constant velocity, by a 90-N force acting at an angle of 15° below the horizontal. (a) How much work is done by the 90-N force in moving the box a distance of 20 m? (b) How much work is done by friction over the same distance? (c) What is the force of friction?

5.3 A force of 20 N parallel to a 37° plane pulls a 2-kg block 5 m up the plane at a constant speed. (a) How much work has been done by the 20-N force? (b) How much work has been done by friction? (c) How much work has been done by the

ational force acting on the block? (d) What can you say
·the total work done?

nswer: (a) 100 J, (b) −41 J, (c) −59 J, (d) zero.

In Fig. 5-10 the force acting on a body for $x < 10$ m is $0.2x$ N and the force for $x > 10$ m is constant at $F_x =$
Vhat is the work done in going from $x = 0$ to $x = 15$ m?

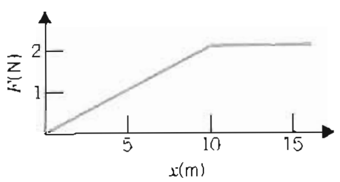


FIGURE 5-10 Problem 5.4.

A boy pulls a 15-kg sled at constant speed a distance of
ilong rough level snow that has a coefficient of friction
. How much work did he do?

A 40-kg box is to be pushed at constant speed a distance
up a ramp by a force parallel to the ramp. The coefficient
tion between the box and the ramp is 0.25. The ramp
an angle of 37° with the horizontal. How much work
be done?

Answer: 1571 J.

A car traveling at 30 m/sec suddenly brakes. If the
ient of friction between the tires and the road is 0.7,
is the minimum stopping distance? Solve by energy
ds.

A bead having an initial speed at point A of 2 m/sec
down a frictionless wire (see Fig. 5-11). What are its
at points B and C?

Answer: $v_B = 4.21$ m/sec, $v_C = 3.44$ m/sec.

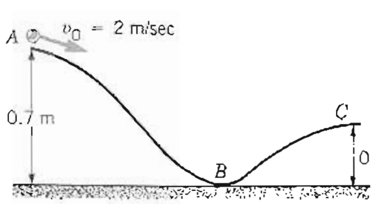


FIGURE 5-11 Problem 5.8.

A horizontal force of 20 N pulls a 10-kg block on a level
nless surface a distance of 5 m. (a) How much work is
done, and what becomes of this work? (b) Show, using the methods of Chapters 3 and 4, that the change in the kinetic energy is equal the work done by the force.

5.10 An automobile is moving with a velocity of 90 km/h. From what height would it have to fall to acquire that velocity?

5.11 Using the conservation of energy principle, find the maximum height reached by a projectile launched with a velocity of 80 m/sec at an angle of 37° with the horizontal?

5.12 A pendulum consists of a mass at the end of a string 1.5 m long. The mass is pulled sideways until the string makes an angle of 30° with the vertical; then it is released. What is the speed of the mass as it passes through its lowest point?

Answer: 1.98 m/sec.

5.13 A light rope passing over a frictionless pulley connects two blocks of mass $m_1 = 3$ kg and $m_2 = 5$ kg (see Fig. 5.12). (a) If the blocks are released from rest in the position shown in Fig. 5-12, what will be the velocity of m_2 as it hits the ground? (b) What is final height reached by m_1?

Answer: (a) 5.42 m/sec, (b) 7.5 m.

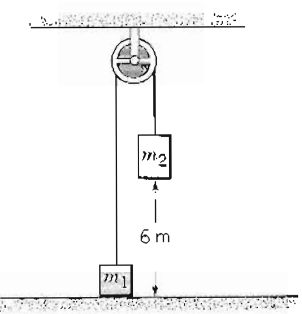


FIGURE 5-12 Problem 5.13.

5.14 A 40-kg box slides 5 m down a ramp inclined at 37° to the horizontal. If there is no friction, what is its speed at the bottom? If the coefficient of friction is 0.2, what is its speed at the bottom?

5.15 A 5-kg block rests at the top of a rough plank inclined at 25° with respect to the horizontal and 4 m long. It is given an initial speed downward of 2 m/sec, and it just reaches the bottom before it stops. What is the coefficient of friction? Solve by energy methods.

Answer: 0.52.

5.16 A con
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incline is 0.25
point if the bl
at that point,
go? (c) At the
What is its spe
methods.

Answer: (a

5.17 A rigi
has a mass of
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As the mass sw
force of 12 N
(a) What is th
speed of the m

5.18 A mass
by a light string
mass $m_2 = 5$ kg
The coefficient
The blocks are

5.16 A constant force of 60 N parallel to an inclined plane of 30° above the horizontal pushes a 6-kg block 10 m up the incline. The coefficient of friction between the block and the incline is 0.25. (a) What is the velocity of the block at the 10-m point if the block starts from rest? (b) If the force is removed at that point, how much farther up the incline will the block go? (c) At the uppermost point the block starts sliding down. What is its speed when it reaches the bottom? Use the energy methods.

Answer: (a) 7.72 m/sec, (b) 4.24 m, (c) 8.90 m/sec.

5.17 A rigid rod of negligible weight and length $l = 2$ m has a mass of 5 kg attached to one end. The other end is pivoted about a point 0, as shown in Fig. 5-13. The mass is released from the position shown with some initial speed v_0. As the mass swings around it experiences an average frictional force of 12 N and just reaches the top of the circle, point P. (a) What is the initial speed v_0 of the mass? (b) What is the speed of the mass as it passes through the lowest point P'?

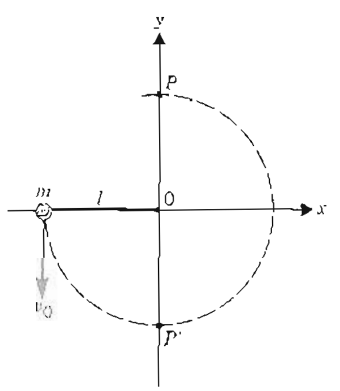


FIGURE 5-13 Problem 5.17.

5.18 A mass $m_1 = 3$ kg resting on a long table is connected by a light string passing over a frictionless pulley to a second mass $m_2 = 5$ kg hanging 2 m above the floor (see Fig. 5-14). The coefficient of friction between the table and m_1 is 0.3. The blocks are released from rest. (a) What is the velocity of m_2 as it hits the floor? (b) What is the total distance traveled by m_1 before it stops?

Answer: (a) 4.48 m/sec, (b) 5.42 m.

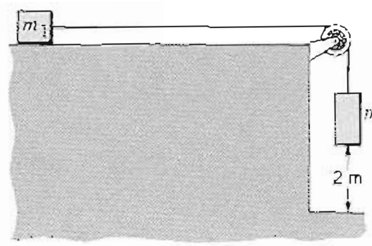


FIGURE 5-14 Problem 5.18.

5.19 An 80-kg man ascends a 4 m high staircase in 12 sec. What is his horsepower?

5.20 An elevator of mass 800 kg is raised 10 m in 5 sec. How much power is required? Express your answer in watts and in horsepower.

5.21 A cable car is operated on a slope 1000 m long making an angle of 20° with the horizontal. The cable car moves up the slope with a speed of 3 m/sec and carries 20 persons of average weight 600 N. What power is needed?

Answer: 1.23×10^4 W.

5.22 The piston of a steam engine is driven 120 times per minute. The length of the stroke is 0.5 m. If the engine develops 150 kW of power, what is the average force exerted by the steam on the piston?

5.23 A pump is needed to lift 100 kg of water per minute from a well 30 m deep. The water is ejected with a speed of 5 m/sec. What must be the power output of the pump?

Answer: 510.8 W.

5.24 A 2500-kg automobile develops 30 kW of power to drive with a constant velocity of 90 km/in on a level road. What power must it develop to drive up a 15° hill with the same velocity?

Answer: 1.88×10^5 W.

6.1 INTRODUCTION

Momentum is the product of the mass of a body and its velocity. It is therefore a vector. We first mentioned momentum in Chapter 4 as the seed whose germination led to modern thoughts on the mechanical behavior of newtonian bodies. In our description of this development, however, we did not consider with sufficient care what was meant by a body. In many cases it is an assembly of particles. In this chapter we will first show how such an assembly can be mathematically represented by a point mass, called the *center of mass*. We will then show that the motion of the center of mass is that predicted by Newton's second law for a particle whose mass is the sum of the masses of the individual particles and is acted on by the resultant of the forces acting on the body. Having established these facts, we will turn our attention to the momentum changes of colliding bodies with confidence, knowing that the treatment of bodies is as mathematically sound as if they were very small masses. Collision theory is very important in our later analysis of conduction electrons in solids.

6.2 CENTER OF MASS

If you have a stick, whether uniform or not, you can find a point along its length that we call the "balance point." If you place your finger there, you can support the stick. Clearly, from Newton's law, the sum of the forces in the y direction is zero at that point, with your finger supplying the upward force. The weight of the stick supplies the downward force, and it appears to be located at that point, although we know that every segment of the stick has weight. We call this point the *center of gravity* of the stick.

If, while balancing the stick on your finger, you suddenly exert an impulse on it by moving your finger rapidly upward, the stick will fly upward without rotating. That is, all parts of the stick will move upward uniformly and the stick will therefore retain the configuration that it had on your finger. If, when it is in motion, you catch it at the center of gravity, the entire stick will stop. These experiments show that the center of gravity of the stick behaves as a point mass in Newton's second law, $\mathbf{F} = m\mathbf{a}$, and that the center of gravity may also be considered as the *center of mass*. That is, if you perform these same experiments in distant space where the force of gravity is negligible, you will obtain the same results. Similarly, if you have a piece of cardboard, you can find a point on the surface at which you can place your finger and support it. All the weight of the cardboard can be considered to be located at that point. Although it is a more difficult experiment to perform on a three-dimensional object, it may be shown mathematically that it too has a center of mass. A better understanding of what the center of mass is will be obtained when we introduce the concept of torque in Chapter 8.

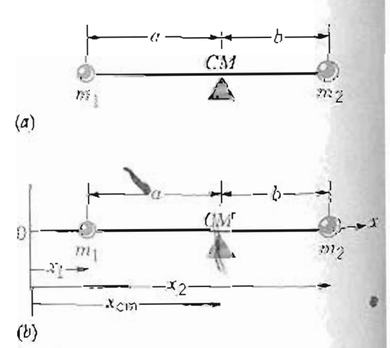


FIGURE 6-1

Suppose that we have two masses, m_1 and m_2, on a weightless stick as in Fig. 6-1*a*. If we perform a measurement, we will find that the balance point (center of mass), is the point at which the products of the masses and their respective distances from the balance point are equal. From Fig. 6-1*a*, the center of mass is the point such that

$$m_1 a = m_2 b \tag{6.1}$$

Let us put this figure on an x axis, as in Fig. 6-1*b*. We may express Eq. 6.1 in terms of x distances by noting that

$$a = x_{cm} - x_1 \quad \text{and} \quad b = x_2 - x_{cm}$$

Equation 6.1 becomes

$$m_1(x_{cm} - x_1) = m_2(x_2 - x_{cm})$$

Rearranging gives

$$(m_1 + m_2)x_{cm} = m_1 x_1 + m_2 x_2$$

or

$$x_{cm}\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

This relation holds true regardless of the number of masses placed on the balance, so we may write

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \tag{6.2}$$

But $\sum_{i=1}^{n} m_i = M$, where M is the total mass. We can therefore express Eq. 6.2 as

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \tag{6.3}$$

EXAMPLE 6-1

Find the center of mass of the configuration in Fig. 6-2 when $m_1 = 1$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg.

Solution

$$x_{cm} = \frac{1}{M}\sum m_i x_i$$

If we take the position of m_1 as the origin, we write this equation as

$$x_{cm} = \frac{1}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}}(1\,\text{kg} \times 0\,\text{m} + 2\,\text{kg} \times 0.5\,\text{m} + 3\,\text{kg} \times 1.3\,\text{m})$$
$$= 0.82\,\text{m}$$

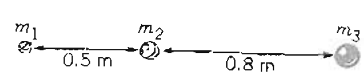


FIGURE 6-2 Example 6-1.

If we had chosen any other point as the origin, the position of the center of mass relative to the individual masses would have been the same, although the numerical value of x_{cm} would have been different.

In the general case, when the masses do not lie on one of the axes, we define the center of mass as the point whose cartesian coordinates are

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \tag{6.3}$$

$$y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \tag{6.4}$$

$$z_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i \tag{6.5}$$

where x_i, y_i, z_i are the coordinates of the ith particle, all measured from the same arbitrary origin. The reason for defining the center of mass in this manner will become obvious in the next section.

EXAMPLE 6-2 Find the x and y coordinates of the center of mass of the system shown in Fig. 6-3, where $m_1 = 2$ kg, $m_2 = 3$ kg, $m_3 = 4$ kg, and $m_4 = 1$ kg, and the coordinates are (3,4)m, (4,6)m, (5,5)m, and (6,8)m, respectively.

Solution We note that each mass has both an x and y coordinate and therefore each contributes to the x_{cm} and the y_{cm}.

For x_{cm} we write

$$\begin{aligned} x_{cm} &= \frac{1}{M}\sum_{i=1}^{n} m_i x_i \\ &= \frac{1}{10\,\text{kg}}(2\,\text{kg} \times 3\,\text{m} + 3\,\text{kg} \times 4\,\text{m} + 4\,\text{kg} \times 5\,\text{m} + 1\,\text{kg} \times 6\,\text{m}) \\ &= 4.4\,\text{m} \end{aligned}$$

For y_{cm} we write

$$\begin{aligned} y_{cm} &= \frac{1}{M}\sum_{i=1}^{n} m_i y_i \\ &= \frac{1}{10\,\text{kg}}(2\,\text{kg} \times 4\,\text{m} + 3\,\text{kg} \times 6\,\text{m} + 4\,\text{kg} \times 5\,\text{m} + 1\,\text{kg} \times 8\,\text{m}) \\ &= 5.4\,\text{m} \end{aligned}$$

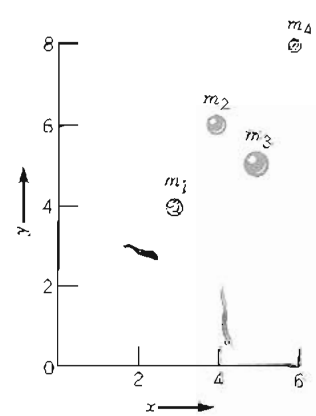


FIGURE 6-3 Example 6-2.

6.3 MOTION OF THE CENTER OF MASS

Let us rewrite the expression for the x coordinate of the center of mass of an array of masses, Eq. 6.3, in the following way.

$$Mx_{cm} = m_1x_1 + m_2x_2 + \cdots m_nx_n \tag{6.6}$$

Differentiating this with respect to time, we obtain

$$M\frac{dx_{cm}}{dt} = m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} + \cdots m_n\frac{dx_n}{dt}$$

or

$$Mv_{xcm} = m_1v_{x1} + m_2v_{x2} + \cdots m_nv_{xn} \tag{6.7}$$

Similar expressions can be readily obtained for Mv_{ycm} and Mv_{zcm}.

Equation 6.7 and the equivalent equations for the y and z motion show that the total momentum of all the particles is equal to the momentum of a single particle whose mass is equal to the sum of the masses of the particles and moves with the velocity of the center of mass. Let us now differentiate Eq. 6.7 with respect to time.

$$M\frac{dv_{xcm}}{dt} = m_1\frac{dv_{x1}}{dt} + m_2\frac{dv_{x2}}{dt} + \cdots m_n\frac{dv_{xn}}{dt}$$

or

$$M\,a_{xcm} = m_1\,a_{x1} + m_2\,a_{x2} + \cdots m_n\,a_{xn} \tag{6.8}$$

We can apply Newton's second law ($\mathbf{F} = m\mathbf{a}$) to each individual particle; that is, $F_{x1} = m_1a_{x1}$, $F_{x2} = m_2a_{x2}$, ... Substituting this in Eq. 6.8, we have

$$M\,a_{xcm} = F_{x1} + F_{x2} + \cdots F_{xn} \tag{6.9}$$

where F_{xi} is the x component of the resultant (i.e., the sum) of the forces acting on the ith particle. A system of masses may be connected by internal forces such as the binding forces of a solid. There may be additional external forces acting on the solid. By Newton's third law of action and reaction, each force exerted internally by a particle on another has an equal and opposite force exerted internally on it. Therefore, the sum of internal forces on the right side of Eq. 6.9 must be zero. We conclude that the sum of the forces in Eq. 6.9 includes only the *external* forces acting on the system of particles. We can rewrite Eq. 6.9 simply as

$$\sum_{i=1}^{n} F_{xi} = Ma_{xcm}$$

The same equation can be derived for the y and the z components of the external forces, and we have the vector equation of Chapter 4

$$\mathbf{F} = M\mathbf{a}_{cm} \tag{6.10}$$

where F is the resultant of the external forces acting on all the particles. This result shows that the center of mass moves as if it were a point whose mass is equal to the total mass of the system and all the external forces were acting on it. And this is why the point defined by Eqs. 6.3, 6.4, and 6.5 is called the center of mass.

EXAMPLE 6-3

Suppose a grenade is thrown that has the trajectory shown in Fig. 6-4. If it explodes in midair, only internal forces have acted on the fragments and therefore from Eq. 6.10 the acceleration of the center of mass of the fragments, regardless of their subsequent dispersal, is unchanged by the explosion, and thus follows the original trajectory.

6.4 MOMENTUM AND ITS CONSERVATION

Recall Newton's approach to mechanics from Chapter 4. He said that an impulse applied to a body will change its state of momentum (Eq. 4.1).

$$\mathrm{F}\Delta t = \Delta m\mathrm{v} \tag{4.1}$$

$$\mathrm{F}\Delta t = m\mathrm{v}_f - m\mathrm{v}_0$$

where v_0 is the velocity of the body before the force begins to act on it and v_f is the velocity when the force stops acting on the body.

Momentum is often represented by the letter p,

$$\mathrm{F}\,\Delta t = \mathrm{p}_f - \mathrm{p}_0 \tag{4.1'}$$

If there is no external force exerted on a mass, the left side of Eq. 4.1′ is zero and we may write

$$\mathrm{p}_0 = \mathrm{p}_f \tag{6.11}$$

This simple equation is called the law of *conservation of momentum*. It is important to recognize that momentum is a vector and that Eq. 6.11 must be satisfied in all three cartesian coordinates.

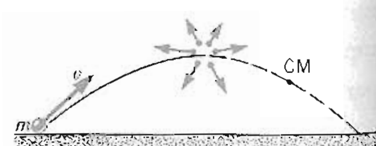


FIGURE 6-4 A grenade explodes while in a trajectory of projectile motion. Because there has been no external force involved in the explosion, the motion of the center of mass is unchanged.

We can easily extend this law to a system of particles using the results developed in the preceding section.

If the resultant of the external forces acting on all the particles is zero, then from Eq. 6.10 the acceleration of the center of mass is $a_{cm} = 0$. Therefore, the velocity of the center of mass will be constant. We can then conclude, from Eq. 6.7 and the equivalent equations for the y and z directions, that the total momentum of all the particles will not change, or

$$\left(\sum_{i=1}^{n} \mathbf{p}_i\right)_{\text{before}} = \left(\sum_{i=1}^{n} \mathbf{p}_i\right)_{\text{after}} \tag{6.12}$$

We should note that Eq. 6.12 does not imply that the momenta of the individual particles remains constant. The individual momenta can change as a result of internal forces such as in Example 6-3, but the total momentum remains unchanged.

EXAMPLE 6-4

A cannon of mass 1000 kg fires a 100-kg projectile with a muzzle velocity of 400 m/sec (see Fig. 6-5). With what speed and in what direction does the cannon move?

Solution Let M be the mass of the cannon, $\mathbf{V}_0$ its initial velocity, and $\mathbf{V}_f$ its final velocity. Let m be the mass of the projectile and $\mathbf{v}_0$ and $\mathbf{v}_f$ its initial and final velocities, respectively. If we consider the cannon and projectile as our system of particles, no external force is involved in the firing of the projectile and we conclude that

$$\mathbf{p}_0 = \mathbf{p}_f$$

Substitute the terms on each side of this equation

$$m\mathbf{v}_0 + M\mathbf{V}_0 = m\mathbf{v}_f + M\mathbf{V}_f$$

If we choose the direction of motion of the projectile as the positive direction, and noting that $\mathbf{V}_0$ and $\mathbf{v}_0$ are zero, we get

$$0 + 0 = m\mathbf{v}_f + M\mathbf{V}_f$$

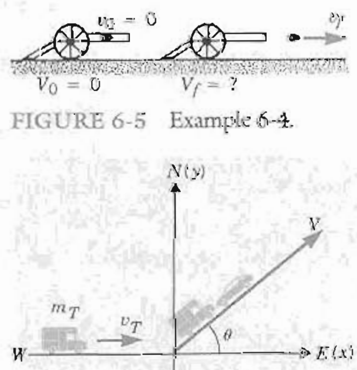


FIGURE 6-5 Example 6-4.

or

$$\mathbf{V}_f = -\frac{m\mathbf{v}_f}{M} = -\frac{100\,\text{kg} \times 400\,\text{m/sec}}{1000\,\text{kg}}$$

$$\mathbf{V}_f = -40\,\text{m/sec}$$

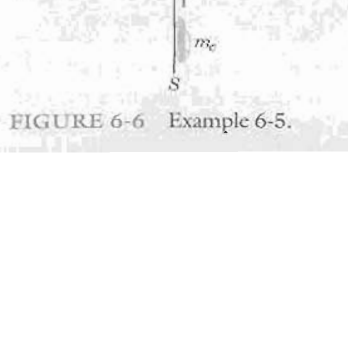


FIGURE 6-6 Example 6-5.

Note that although from experience we know that the cannon will recoil (i.e., move in a direction opposite to that of the projectile), we are not told this as one of the facts of the problem. So we put $\mathbf{V}_f$ in as positive; the vector aspect of the formulation shows in the result (i.e., the fact that $\mathbf{V}_f$ is negative) that the direction of recoil is opposite to that of the projectile.

In the next section we will consider problems in collisions whose solutions involve both momentum conservation and energy accountability. However, as the following example illustrates, some questions about collisions can be answered by momentum conservation alone.

EXAMPLE 6-5

A 10,000-kg truck traveling east at 20 m/sec collides with a 2000-kg car traveling north at 30 m/sec. After the collision, they are locked together. With what velocity and at what angle do the locked vehicles move immediately after the collision? (See the schematic diagram, Fig. 6-6.)

Solution Because no external force is involved in the collision, momentum is conserved. In the x direction

$$p_{x0} = p_{xf}$$

$$m_T v_T = (m_T + m_c)V \cos\theta$$

Rearranging terms,

$$V \cos\theta = \frac{m_T v_T}{m_T + m_c} = \frac{10,000 \text{ kg} \times 20 \text{ m/sec}}{10,000 \text{ kg} + 2000 \text{ kg}}$$

$$= 16.7 \text{ m/sec}$$

In the y direction,

$$p_{y0} = p_{yf}$$

$$m_c v_c = (m_T + m_c)V \sin\theta$$

Solving for $V \sin\theta$,

$$V \sin\theta = \frac{m_c v_c}{m_T + m_c} = \frac{2000 \text{ kg} \times 30 \text{ m/sec}}{10,000 \text{ kg} + 2000 \text{ kg}}$$

$$= 5.0 \text{ m/sec}$$

First find the angle by dividing the two velocity components

$$\frac{\not{V} \sin\theta}{\not{V} \cos\theta} = \tan\theta = \frac{5.0 \text{ /sec}}{16.7 \text{ m/sec}} = 0.30$$

$$\theta = \arctan 0.30 = 16.7^\circ$$

Find V by substituting the angle into either the x or y momentum solutions

$$V \sin 16.7^\circ = 5 \text{ m/sec}$$

$$V = 17.4 \text{ m/sec}$$

or

$$V \cos 16.7^\circ = 16.7 \text{ m/sec}$$

$$V = 17.4 \text{ m/sec}$$

6.5 COLLISIONS

One of the most important applications of the conservation of momentum law occurs in the theory of collisions. We will deal only with collisions between two bodies, as it is exceedingly difficult to obtain any but approximate solutions for three-body collisions. There are two types of collisions, to which we give the names *elastic* and *inelastic*. In an elastic collision kinetic energy is conserved (i.e., no energy is lost from the system). This type of collision can occur only between atomic particles, although in physics problems we often assume elastic collisions between colliding bodies. In actuality, there are no elastic collisions, but in some the energy loss is very small and it may be considered negligible. An inelastic collision is one in which kinetic energy is not conserved (e.g., some energy is lost to friction, crumpled fenders, or such).

6.5a. Elastic Collisions

EXAMPLE 6-6

A neutron with a mass of $m = 1\,\text{u}$ (atomic mass unit) strikes a larger atom at rest and rebounds elastically along its original path with 0.9 of its initial forward velocity. What is the mass M, in atomic mass units, of the atom it struck?

Solution Let v_0 be the initial velocity of the neutron and $v_f = -0.9v_0$ its final velocity. Note that the problem tells us that it rebounds; therefore the direction of the final velocity is opposite to its initial velocity. Let M be the mass of the atom, V_0 its initial velocity, and V_f its velocity after collision. Both momentum and kinetic energy are conserved. From the conservation of momentum

$$mv_0 + MV_0 = mv_f + MV_f$$

and on rearranging and using the fact that $V_0 = 0$

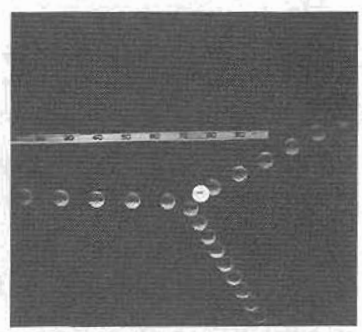
$$V_f = \frac{m(v_0 - v_f)}{M} = \frac{1\,\text{u}(v_0 + 0.9v_0)}{M} = \frac{(1\,\text{u})(1.9v_0)}{M}$$

From the conservation of kinetic energy

$$\frac{1}{2}mv_0^2 + \frac{1}{2}MV_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2$$

Solving for M, noting that $V_0 = 0$, we obtain

$$M = \frac{m(v_0^2 - v_f^2)}{V_f^2} = \frac{(1\,\text{u})(0.19v_0^2)}{V_f^2}$$



Multiflash photograph of a collision between a moving ball coming in from the left and a stationary ball.

If we substitute for v_f from the momentum equation into the energy equation, we obtain

$$M = \frac{(1\,\mathrm{u})(0.19v_0^2)M^2}{(1\,\mathrm{u}^2)(3.61v_0^2)}$$

which simplifies to

$$M = \frac{(1\,\mathrm{u}^2)(3.61)}{(1\,\mathrm{u})(0.19)}$$
$$= 19\,\mathrm{u}$$

EXAMPLE 6-7

An important type of elastic collision at the atomic level, whose results we will use later, is the collision between a very small mass particle, such as an electron, with another particle of comparatively large mass, such as an atom. The mass of a copper atom, for example, is about 10^5 times that of an electron. In this type of collision one is often interested in finding the velocity of the electron after the collision with the copper atom. To solve this type of collision, we follow the usual procedure of conserving momentum and kinetic energy. We will assume a one-dimensional collision.

Solution Let m, v_0, and v_f be the mass and the initial and the final velocity of the electron and M, V_0, and V_f those of the atom. From the conservation of momentum law

$$mv_0 + MV_0 = mv_f + MV_f$$

and, on rearranging,

$$M(V_0 - V_f) = m(v_f - v_0) \tag{6.13}$$

Conserving kinetic energy yields

$$\frac{1}{2}mv_0^2 + \frac{1}{2}MV_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2$$

or

$$M(V_0^2 - V_f^2) = m(v_f^2 - v_0^2)$$

On factoring,

$$M(V_0 + V_f)(V_0 - V_f) = m(v_f + v_0)(v_f - v_0) \tag{6.14}$$

Dividing Eq. 6.14 by Eq. 6.13 obtains

$$V_0 + V_f = v_0 + v_f \tag{6.15}$$

For simplicity, let us have the atom initially at rest, $v_0 = 0$, and substitute v_f of Eq. 6.15 into Eq. 6.13,

$$-M\,(v_0 + v_f) = m\,(v_f - v_0)$$

$$-Mv_0 - Mv_f = mv_f - mv_0$$

Rearranging,

$$-v(M + m) = v_0(M - m)$$

$$v_f = -\,v_0\frac{(M - m)}{(M + m)} \qquad (6.16)$$

For our case $m \ll M$; therefore Eq. 6.16 reduces to

$$v_f \approx -v_0 \qquad (6.17)$$

The electron rebounds (recoils) with the same magnitude of velocity; thus it does not lose any kinetic energy; therefore the atom does not gain any and is not set in motion. This is, of course, an approximation. If the mass of the electron is taken as 1 unit and the mass of the atom as 10^5 units and these numbers are substituted into Eq. 6.16, then $v_f = -0.99998v_0$. Because kinetic energy is proportional to v^2, the remaining energy of the recoiling electron is 0.99996 of its initial kinetic energy. Therefore, in the collision 0.00004 of the initial kinetic energy of the electron has been transferred to the atom. This concept will be important later when we develop the loss of electron energy to atoms in an electrical conductor in which electrons flow as a current. The increase in energy of the atoms from electron collisions manifests itself as an increase in temperature of the conductor.

6.5b. Inelastic Collisions

In all the preceding examples, the energy of the system was unchanged by the collision. We now give an example of a collision in which the energy is changed by the collision.

EXAMPLE 6-8

A ballistic pendulum is used to measure the velocity of a bullet. The bullet is shot into a wooden block suspended by strings. It lodges in the block, losing energy in its penetration, and the increase in the height of the swinging block and bullet is measured (see Fig. 6-7). If the bullet has a mass of 0.01 kg, the block has a mass of 0.5 kg, and the swing rises 0.1 m, what was the velocity of the incident bullet and what fraction of its energy was lost during penetration?

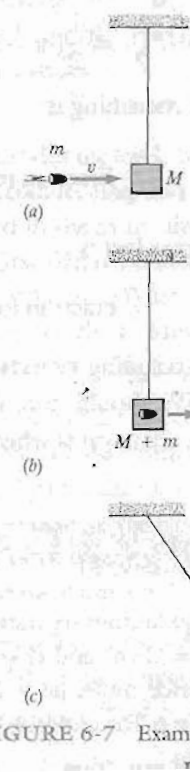


FIGURE 6-7 Example 6-8.

Solution We first conserve momentum between situation (a) and (b) in Fig. 6-7.

$$p_0 = p_f$$

$$mv + 0 = (m + M)V$$

$$0.01\,\text{kg}\,v = 0.51\,\text{kg}\,V$$

We saw in Section 5.6 in the discussion of the pendulum that the string does no work on the block. We may therefore conserve mechanical energy between situations (b) and (c) in Fig. 6-7.

$$(E_k)_b = (E_p)_c$$

$$\frac{1}{2}(m + M)V^2 = (m + M)gh$$

$$V = \sqrt{2gh} = 1.4\,\text{m/sec}$$

Substitute this value into the momentum equation and obtain

$$v = \frac{0.51\,\text{kg}}{0.01\,\text{kg}} \times 1.4\,\text{m/sec} = 71.4\,\text{m/sec}$$

We find the fraction of the bullet's initial energy lost in penetration by calculating the energy of the system before (situation a) and after the collision (situation b).

$$(E_k)_a = \frac{1}{2}mv^2 = \frac{1}{2}(0.01\,\text{kg})(71.4\,\text{m/sec})^2 = 25.5\,\text{J}$$

$$(E_k)_b = \frac{1}{2}(m + M)V^2 = \frac{1}{2}(0.51\,\text{kg})(1.4\,\text{m/sec})^2 = 0.5\,\text{J}$$

The fraction remaining is

$$\text{Fraction remaining} = \frac{(E_k)_b}{(E_k)_a} = \frac{0.5\,\text{J}}{25.5\,\text{J}} = 0.02$$

and the fraction lost is

$$\text{Fraction lost} = 1 - \text{fraction remaining} = 0.98$$

When colliding objects stick together we find that the kinetic energy is not conserved. We should note that energy can be lost, and therefore the collision is inelastic, in certain cases where objects do not stick together.

PROBLEMS

6.1 The equilibrium separation between the centers of the sodium ($m = 23\,\text{u}$) and chlorine ($m = 35\,\text{u}$) ions in the sodium chloride molecule is 2.4×10^{-10} m. Where is the center of mass of the molecule?

6.2 In the Bohr model of the hydrogen atom, the electron ($m = 9.1 \times 10^{-31}$ kg) revolves around a proton ($m = 1.67 \times 10^{-27}$ kg) in a circular orbit of radius $r = 0.5 \times 10^{-10}$ m. Where is the center of mass of the hydrogen atom?

6.3 What are the x and y coordinates of the center of mass of the system of particles shown in Fig. 6-8?

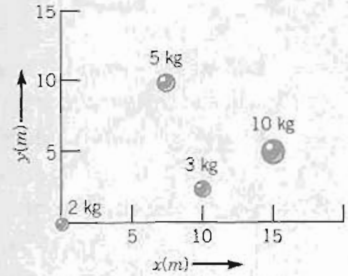


FIGURE 6-8 Problem 6.3.

6.4 A thorium nucleus ($m = 232$ u), at rest at the origin of a coordinate system, decays into a radium nucleus ($m = 228$ u) and an alpha particle ($m = 4$ u). Sometime later, the alpha particle passes the point $x = 3$ m, $y = 2$ m with a velocity $v = 2 \times 10^6$ m/sec. What is the position and the velocity of the radium nucleus at that moment?

Answer: $x = 5.3 \times 10^{-2}$ m, $y = 3.5 \times 10^{-2}$ m, $v = 3.5 \times 10^4$ m/sec.

6.5 Two particles of mass $m_1 = 1$ kg and $m_2 = 99$ kg are held 2 m apart. The particles attract each other with a constant force directed along the line joining the two particles. (a) When the particles are released, where will the collision occur? (b) Does the answer to (a) depend on the actual value of the force?

Answer: (a) 1.98 m from m_1, (b) no.

6.6 A 0.25-kg baseball has an initial velocity toward a bat of 15 m/sec. The batter strikes the ball and it goes out in the reverse direction at 30 m/sec. (a) What is the change in the momentum of the ball? (b) What is its change in kinetic energy?

6.7 A swimmer in a pool makes a racing turn at the end by suddenly straightening his legs while his feet are pressed against the end of the pool. If his mass is 80 kg and he exerts an average force of 120 N for 0.8 sec, what is his initial velocity on leaving the pool end?

6.8 A car crashes into a tree. If the car has a mass of 1200 kg and its speed is reduced from 30 m/sec to zero in 0.2 sec, what is the average force exerted by the tree on the car?

electron
= 1.67 ×
10^{-10} m.

6.9 A fire hose delivers water at the rate of 20 kg/sec with a speed of 25 m/sec. A riot police officer uses the hose to control an unruly crowd. The water from the hose strikes a person horizontally and then falls down to the ground. What is the average force experienced by that person?

Answer: 500 N.

6.10 A gun fires a 0.01-kg bullet with a velocity of 250 m/sec at a 0.5-kg melon resting on a post. The bullet penetrates the melon and leaves the back of it with a velocity of 100 m/sec. With what velocity and in what direction does the melon leave the post?

6.11 A radium atom at rest with a mass of 226 u suddenly emits an alpha particle of mass 4 u with a speed of 2×10^7 m/sec. With what speed and in what direction does the resulting radon atom of mass 222 u move?

6.12 A truck of mass 5×10^3 kg moving at 20 m/sec collides head-on with a car of mass 1×10^3 kg moving at 25 m/sec in the opposite direction. If they stick together after the collision, in what direction and with what speed do they move immediately after the collision?

6.13 An atom of mass 10 u strikes a stationary atom of mass M and rebounds elastically with one half its original velocity. What is the mass of the atom it struck?

Answer: 30 u.

6.14 A sled of mass 10 kg slides on level, frictionless ice with a velocity of 12 m/sec. It collides elastically with another sled of different mass pointed in the same direction but at rest. After the collision, the first sled continues in the same direction but with a velocity of 4 m/sec. What is the mass of the second sled and its velocity after the collision?

Answer: 5 kg, 16 m/sec.

6.15 A 9000-N open-top railroad car is coasting with a velocity $v = 10$ km/h on a frictionless horizontal track. A 1200-kg meteorite falls vertically into the car with a velocity of 200 km/h. (a) What is the velocity of the railroad car after the meteorite lands on it? (b) What is the magnitude and the direction of the impulse of the meteorite on the car?

6.16 An object at rest in space explodes into three equal parts. The velocities of two of them are, respectively, $2i$ and $-4j$. Find the resulting velocity of the third part.

6.17 A bullet of mass 80 g is moving east with a velocity v_0. The bullet strikes a 200-g wooden block moving south with a velocity of 2 m/sec. The bullet remains embedded in the block, which then moves in the direction 37° south of east. (a) What is the velocity of the block after the collision? (b) What was the initial velocity v_0 of the bullet? (c) What is the fractional change in the energy of the system?

Answer: (a) 2.37 m/sec, (b) 6.65 m/sec, (c) 64% decrease.

6.18 Two particles of mass $m_1 = 5$ kg and $m_2 = 2$ kg move toward each other as shown in Fig. 6-9. After the collision, they stick together. (a) What is the speed of the particles after they collide? (b) What is the direction of motion of the particles after the collision? (c) What is the change in the total kinetic energy of the particles?

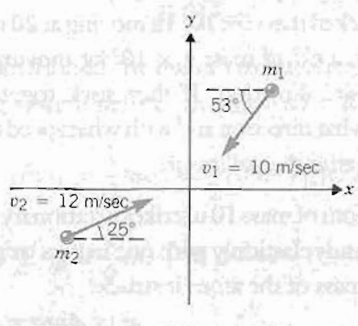


FIGURE 6-9 Problem 6.18.

6.19 A puck sliding on a frictionless table with a velocity $v = 2$ m/sec strikes a second puck of equal mass initially at rest. The collision is elastic, and it is found that after the collision both pucks move with the same speed. (a) What is the speed of the pucks after the collision? (b) What is the angle between the directions of motion of the pucks?

Answer: (a) 1.41 m/sec, (b) 90°.

6.20 Three boys stand on a 10-kg wagon resting on a frictionless horizontal surface. The boys take turns running off the same end of the wagon with a velocity of 1.5 m/sec relative to the wagon. The mass of each of the boys is 40 kg. What is the final velocity of the wagon?

Answer: 7.87 m/sec.

6.21 A 2-kg block rests on the ground. The coefficient of friction between the block and the ground is 0.4. A man fires a 0.01-kg bullet parallel to the ground. It lodges in the block, and the block and bullet are observed to slide 2 m before coming to rest. What was the velocity of the bullet?

Answer: 796 m/sec.

6.22 A block of mass 1 kg rests over a hole in a tabletop. A bullet of mass 0.01 kg is fired upward into the block with a velocity of 200 m/sec. If the bullet imbeds itself in the block, how high will the block rise?

6.23 A bullet of mass 100 g is shot into a 3-kg wooden block resting on an incline plane, as shown in Fig. 6-10. The bullet remains embedded in the block, which then slides down the incline plane 2 m before coming to rest. The coefficient of friction between the block and the incline is 0.5. What was the initial velocity of the bullet?

Answer: 91.9 m/sec.

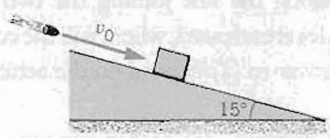


FIGURE 6-10 Problem 6.23.

6.24 A bomb is launched with a velocity $v_0 = 500$ m/sec at an angle of 37° with the horizontal. At the highest point of the trajectory it explodes in two equal pieces. One piece lands 20 sec later directly below the point where the explosion occurred. When and where does the second piece land?

Answer: 77.8 sec, 4.99×10^4 m from launching.

7.1 INTRODUCTION

In the preceding chapters we dealt with translational motion—that is, the change of position in the cartesian coordinate system of the center of mass of a body. However, some systems simply rotate and some rotate while the center of mass translates through space: A roulette wheel simply rotates, whereas a car wheel both rotates and translates. Newton's laws and momentum and energy conservation still apply, but the formulation is somewhat different. In this chapter and the next we will consider rotational motion. We need these properties in order to build the model of the electron rotational motion in atoms.

7.2 MEASUREMENT OF ROTATION

The most common measurement of rotation is a count of the number of revolutions about an axis of rotation. We also use degrees as a measure, where 360° corresponds to one revolution. In physics we mostly use radians for a variety of reasons. One of these reasons is that the formulation affords a quick and easy bridge between linear and rotational motion. Let us examine this.

A measure of an angle in radians is the length of the circular arc subtended by the angle divided by the radius of the circle (see Fig. 7-1). If the length of the arc from a to b is s and r is the radius, then the measure of angle θ in radians is given by

$$\theta(\text{in radians}) = \frac{s}{r} \tag{7.1}$$

Because both s and r are in units of length, the units cancel on the right side and θ is dimensionless. Other measures of θ, such as degrees or revolutions, are also dimensionless. However, the numerical magnitudes of these quantities differ, so we must state the system of measure used. Obviously, we must maintain a consistency of angular measure in a given problem.

Returning to Eq. 7.1, we may ask how many radians there are in a whole revolution. The arc length subtended by a revolution is the circumference, or $2\pi r$. Therefore,

$$\frac{s \text{ of 1 revolution}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi \text{ radians (rad)}$$

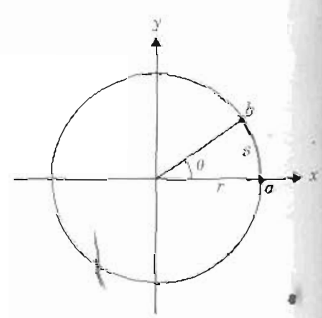


FIGURE 7-1

We may convert between systems of angular measure as shown in Chapter 1 using the identities

$$2\pi \text{ (radians)} = 360 \text{ (degrees)} = 1 \text{ revolution (rev)}$$

7.3 ROTATIONAL MOTION

Suppose we have a reference marker a on the x axis of a coordinate system and a wheel whose center coincides with the origin. We also have a mark b on the wheel. We can measure the time during which the wheel marker b moves from the coordinate marker a, a distance Δs (see Fig. 7-2). The $\text{speed}_{a \to b}$ of the marker on the wheel is measured by the time it takes for the marker to move an arc length Δs, or

$$\text{speed}_{a-b} = \frac{\Delta s}{\Delta t}$$

In the limit $\Delta t \to 0$ the distance Δs becomes a vector and the speed becomes **v**, the instantaneous velocity (see Section 3.2)

$$\mathbf{v} = \frac{ds}{dt} \tag{7.2}$$

The direction of the instantaneous velocity of the marker on the rotating wheel is the tangent to the circle of motion and is called the *tangential* velocity; it is sometimes indicated by writing **v** with the subscript T. Note that as the marker rotates the direction of $\mathbf{v}_T$ constantly changes even though the marker may rotate at a constant rate. Therefore, the vector $\mathbf{v}_T$ is constantly changing. In the previous chapters we have dealt largely with vectors whose direction remained constant while the magnitude changed, whereas here we have a vector whose magnitude may remain constant while its direction always changes. This has important implications in the development of the centripetal force that we will consider later in this chapter.

In Fig. 7-2 we see that while s is increasing, the angle θ is also increasing as the moving radius vector (line from origin to the marker on the circumference) sweeps out a larger arc. The average rate of change of the angle θ with time is called the average *angular* or *rotational velocity*; we use the small Greek letter ω (omega) for this.

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

and, as $\Delta t \to 0$, the average angular velocity $\overline{\omega}$ becomes the instantaneous angular velocity ω, namely,

$$\omega = \frac{d\theta}{dt} \tag{7.3}$$

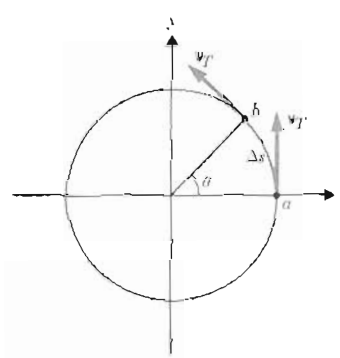


FIGURE 7-2

Because θ is dimensionless, ω has units of reciprocal time, although it must be specified whether ω is radians/second, degrees/second or revolutions/second. We may relate the tangential velocity to the rotational velocity by differentiating Eq. 7.1 with respect to time

$$\frac{1}{r}\frac{ds}{dt} = \frac{d\theta}{dt}$$

Rearranging terms and using the definitions of Eqs. 7.2 and 7.3 we obtain

$$v_T = r\omega \qquad (7.4)$$

where ω is in radians/second.

EXAMPLE 7-1

A car is traveling at a constant velocity of 24 m/sec. The radius of its wheels is $r = 0.30$ m. (a) How many revolutions have the wheels turned after the car has gone 120 m? (b) How many revolutions have the wheels turned after 60 sec?

Solution

- (a) If there is no slipping between the wheels of the car and the road, the arc length moved by a marker on the outermost radius of the wheel is equal to the distance traveled by the car; that is, $s = 120$ m. Using Eq. 7.1

$$\theta = \frac{s}{r} = \frac{120\text{ m}}{0.30\text{ m}}$$

$$\theta = 400\text{ rad} = 400\text{ rad}\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right) = 63.7\text{ rev}$$

- (b) Because the car travels at constant velocity, the distance traveled by the car in 60 sec can be found with Eq. 3.12, keeping in mind that the acceleration $a = 0$

$$x = (24\text{ m/sec})(60\text{ sec}) = 1440\text{ m}$$

This, as we have indicated, is also the arc length moved by a marker on the rim of the wheel. We now use Eq. 7.1 to find the angle rotated by the wheel in 60 sec.

$$\theta(t = 60\text{ sec}) = \frac{s}{r}$$

$$\theta = \frac{1440\text{ m}}{0.30\text{ m}} = 4800\text{ rad} = 4800\text{ rad}\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right) = 764\text{ rev}$$

Suppose the marker on the rotating wheel of Fig. 7-2 is not rotating at a constant rate but is speeding up or slowing down. Then from Eq. 7.4 the marker on the wheel has an average tangential acceleration that from Eq. 3.4 is

$$\bar{a}_T = \frac{\Delta v_T}{\Delta t} = \frac{v_{Tf} - v_{T0}}{\Delta t}$$

In the limit as $\Delta t \to 0$, $\bar{a}_T$ becomes the instantaneous acceleration a_T, that is,

$$a_T = \frac{dv_T}{dt} \qquad (7.5)$$

The rate at which the angle θ is being swept out, the angular velocity ω, is also changing. We call the rate of change of ω the average *angular* or *rotational* acceleration and use as the symbol the small Greek letter α (alpha), so that

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

To find the instantaneous value we let $\Delta t \to 0$, and

$$\alpha = \frac{d\omega}{dt} \tag{7.6}$$

Angular acceleration has dimensions of $(\text{time})^{-2}$, although, as before, we must specify the measure of the angle. We may connect the angular acceleration with the tangential acceleration of the marker on the wheel by differentiating Eq. 7.4 with respect to time.

$$\frac{dv_T}{dt} = r\frac{d\omega}{dt}$$

$$a_T = r\alpha \tag{7.7}$$

where α is in radians/second squared.

Because the radian is a dimensionless quantity, the units of a_T will be the same as those of r divided by second^2. Thus if r is expressed in meters, a_T will be in m/sec^2.

EXAMPLE 7-2

A driver of a car traveling at 24 m/sec applies the brakes, decelerates uniformly, and comes to a stop in 100 m. If the wheels have a radius of 0.30 m, what is the angular deceleration of the wheels in rev/sec^2?

Solution There are several ways to solve this, but let us use the most straightforward one, finding first the linear deceleration and then relating it to rotational deceleration

$$v_0 = 24\ \text{m/sec}, \quad v_f = 0, \quad x = 100\ \text{m}, \quad a = ?$$

From Eq. 3.11,

$$v_f^2 - v_0^2 = 2ax$$

$$a = \frac{v_f^2 - v_0^2}{2x} = \frac{0 - (24\ \text{m/sec})^2}{(2)(100\ \text{m})} = -2.88\ \text{m/sec}^2$$

Because a is the acceleration of the car, it is also the tangential acceleration of every point on the rim of its wheels (assuming no slipping between the wheels and the road). By Eq. 7.7

$$a_T = r\alpha$$

$$\alpha = \frac{a_T}{r} = \frac{-2.88\ \text{m/sec}^2}{0.30\ \text{m}}$$

$$\alpha = -9.6\frac{\text{rad}}{\text{sec}^2}\frac{1\ \text{rev}}{2\pi\ \text{rad}} = -1.5\ \text{rev/sec}^2$$

7.4 EQUATIONS OF ROTATIONAL MOTION

We may derive the equations of rotational motion by the method of Chapter 3. As in Chapter 3, we will limit our discussion to the case of constant angular acceleration.

The basic relations obtained by arguments analogous to those of Eqs. 3.7 and 3.9 are

$$\theta = \overline{\omega} t \tag{7.8}$$

$$\overline{\omega} = \frac{\omega_0 + \omega_f}{2} \tag{7.9}$$

We may obtain the other three relations analogous to Eqs. 3.8, 3.11, and 3.12 by integration for the condition that α = constant. From the definition of α, Eq. 7.6,

$$\alpha = \frac{d\omega}{dt}$$

$$\int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt$$

$$\omega - \omega_0 = \alpha t$$

$$\omega = \omega_0 + \alpha t \tag{7.10}$$

From the definition

$$\omega = \frac{d\theta}{dt}$$

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \omega\, dt$$

Substitute Eq. 7.10 for ω

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)dt = \omega_0 \int_0^t dt + \alpha \int_0^t t\, dt$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 \tag{7.11}$$

To obtain our final equation, we use the chain rule to write

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$$

Multiply both sides by $d\theta$

$$\alpha\, d\theta = \omega\, d\omega$$

and the integration is

$$\alpha \int_{\theta_0}^{\theta} d\theta = \int_{\omega_0}^{\omega} \omega\, d\omega$$

$$\alpha(\theta - \theta_0) = \frac{1}{2}(\omega^2 - \omega_0^2)$$

which is usually written in the form

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0) \qquad (7.12)$$

EXAMPLE 7-3

A roulette wheel is given an initial rotational velocity of 2 rev/sec. It is observed to be rotating at 1.5 rev/sec 5 sec after it was set in motion. (*a*) What is the angular deceleration (assumed constant) of the wheel? (*b*) How long will it take to stop? (*c*) How many revolutions will it make from start to finish?

Solution

(a) $\omega_0 = 2.0\,\text{rev/sec} \quad \omega = 1.5\,\text{sec} \quad t = 5\,\text{sec} \quad \alpha = ?$

$$\omega = \omega_0 + \alpha t$$

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{1.5\,\text{rev/sec} - 2.0\,\text{rev/sec}}{5\,\text{sec}}$$

$$\alpha = -0.1\,\text{rev/sec}^2$$

(b) $\omega_0 = 2\,\text{rev/sec} \quad \omega_f = 0 \quad \alpha = -0.1\,\text{rev/sec}^2 \quad t_f = ?$

$$\omega_f = \omega_0 + \alpha t_f$$

$$t_f = \frac{\omega_f - \omega_0}{\alpha} = \frac{0 - 2\,\text{rev/sec}}{-0.1\,\text{rev/sec}^2}$$

$$t_f = 20\,\text{sec}$$

(c) $\omega_0 = 2\,\text{rev/sec} \quad \omega_f = 0 \quad t_f = 20\,\text{sec} \quad \theta = ?$

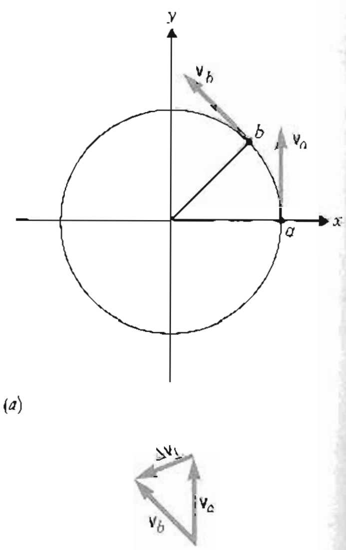
$$\theta = \frac{\omega_0 - \omega_f}{2} t_f = \frac{2\,\text{rev/sec} + 0}{2} \times 20\,\text{sec}$$

$$\theta = 20\,\text{rev}$$

7.5 RADIAL ACCELERATION

Let us consider more carefully the motion of the marker on the wheel in Fig. 7-2 as the wheel rotates at constant speed. In Fig. 7-3*a* we have the same wheel with the velocity vectors indicated at points *a* and *b*. We see that even though the velocity vectors at points *a* and *b* may have the same magnitude, their direction is different. This difference is indicated in Fig. 7-3*b* by the vector $\Delta \mathbf{v}_\perp$. In this figure the vector tails have been put at a common point. Thus, in a time Δt the vector $\mathbf{v}_a$ has changed in value by $\Delta \mathbf{v}_\perp$. This implies an acceleration has taken place. Because a velocity vector of a point on a circle is always tangent to the circle, it is perpendicular to the radius. For infinitesimal changes Δt and thus $\Delta \mathbf{v}_\perp$, there is an acceleration $\mathbf{a}_R$ inward along the radius called a *radial acceleration* $\mathbf{a}_R$ given by

$$\mathbf{a}_R = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$$



(*a*)

(*b*)

FIGURE 7-3

We will now examine this radial acceleration analytically. We see from Fig. 7-4*a* by the method of vector components of Chapter 2 that the coordinates of the marker at some time *t* are

$$x = r\cos\theta$$
$$y = r\sin\theta \tag{7.13}$$

Let the marker rotate about the circle at a constant rotational velocity ω so that $\bar{\omega} = \omega$. Substitute Eq. 7.8 into Eqs. 7.13 and obtain

$$x = r\cos\omega t$$
$$y = r\sin\omega t \tag{7.14}$$

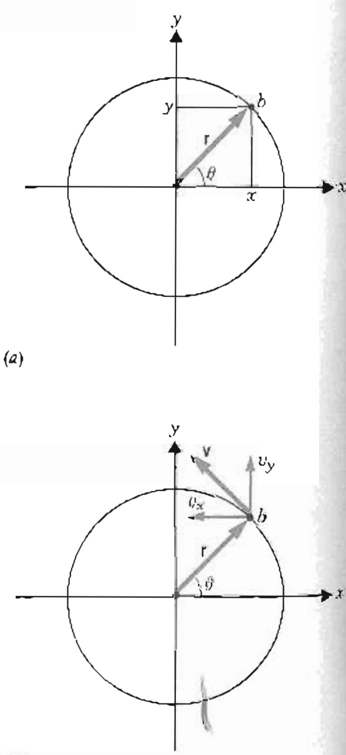
The *x* component of the velocity of the marker in Fig. 7-4*b* is $v_x = dx/dt$, and the *y* component is $v_y = dy/dt$. Performing this differentiation of Eqs. 7.14 yields

$$v_x = r\frac{d}{dt}(\cos\omega t) = -r\omega\sin\omega t$$
$$v_y = r\frac{d}{dt}(\sin\omega t) = r\omega\cos\omega t \tag{7.15}$$

We see that in Eqs. 7.15 both v_x and v_y are functions of time and therefore the point must be accelerating in both the *x* and *y* directions. We may obtain the components of acceleration by differentiating Eqs. 7.15 with respect to time:

$$a_x = \frac{dv_x}{dt} = -r\omega\frac{d}{dt}(\sin\omega t) = -r\omega^2\cos\omega t$$

$$a_y = \frac{dv_y}{dt} = r\omega\frac{d}{dt}(\cos\omega t) = -r\omega^2\sin\omega t \tag{7.16}$$



(*a*)

(*b*)

FIGURE 7-4*a* and *b*

The square of the resultant acceleration a_R^2 is the sum of the squares of the components, or

$$a_R^2 = a_x^2 + a_y^2$$

$$a_R^2 = r^2\omega^4 \cos^2 \omega t + r^2\omega^4 \sin^2 \omega t$$

$$= r^2\omega^4(\cos^2 \omega t + \sin^2 \omega t)$$

Using the trigonometric identity that

$$\sin^2 \theta + \cos^2 \theta = 1$$

we obtain

$$a_R^2 = r^2\omega^4$$

or

$$a_R = \pm r\omega^2 \qquad (7.17)$$

The direction of $\mathbf{a}_R$ can be found by comparing Eq. 7.16 with Eq. 7.14. It is seen that a_x is ω^2 times the *negative x* coordinate of the radius vector **r** and a_y is ω^2 times the *negative y* coordinate of **r**. This implies that $\mathbf{a}_R$, and, consequently, the direction of $\mathbf{a}_R$ is along the radius toward the center, that is, opposite to the vector direction of the radius. This is sketched in Fig. 7-5, where the resultant $\mathbf{a}_R$ is seen to be directed inward along the radius *toward* the center.

7.6 CENTRIPETAL FORCE

Newton's second law, $\mathbf{F} = m\mathbf{a}$, states that if there is a net force on a body there is an associated acceleration. The converse is true; if there is an acceleration there must be a net force. We have shown in the previous section that a particle or a body moving in circular motion at constant speed is being accelerated inward along the radius. Therefore, the particle must be acted on by a force along the radius toward the center. This situation corresponds to the statement of Newton's second law that "If a body in a state of motion is acted on by an external force, it will be accelerated in the direction of the force." The particle *cannot* undergo circular motion unless there is a force along the radius directed inward toward the center. This force is called the *centripetal* (center-seeking) force. A demonstration of this is easily performed by whirling a weight at the end of a string in a circle. You must exert a constant force (tension in the string) to maintain the motion. If you let go of the string, the weight will fly off in a straight line with a velocity whose direction will be the tangent to the circle at the point of release.

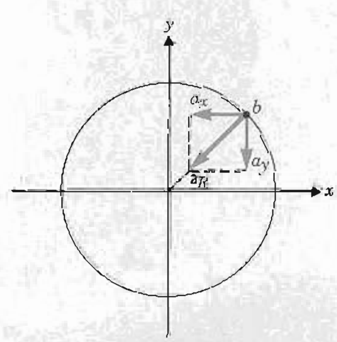


FIGURE 7-5

We indicate radial (or centripetal) force by F_R. We may use Newton's second law to write

$$\sum F_R = ma_R \tag{7.18}$$

or, using Eq. 7.17

$$\sum F_R = mr\omega^2 \tag{7.19}$$

Another convenient form is obtained by substituting Eq. 7.4, $v_T = r\omega$ for ω

$$\sum F_R = m\frac{v_T^2}{r} \tag{7.20}$$

In the solution of problems involving radial acceleration, two rules must be observed, based on the derivations: (1) a_R has dimensions of m/sec^2 and therefore ω must have dimensions of rad/sec^2; and (2) radial forces directed toward the center of rotation are positive, whereas those directed away from the center are negative. We also note from Section 3-2 that the magnitude of the instantaneous tangential velocity at any point is equal to the speed.

EXAMPLE 7-4

A person whose weight is 600 N is riding a roller coaster. This person sits on a scale as the roller coaster passes over the top of a rise of radius 80 m. (a) What is the minimum speed of the car if the scale reads zero (the sensation of weightlessness is experienced)? (b) If the car increases its speed to 40 m/sec in descending to a dip with a radius of 80 m, what will the scale read? See Fig. 7-6.

Solution Let us consider the forces on the rider at the rise. The rider's weight, mg, is directed toward the center. The scale exerts a normal force N upward. N is the reading of the scale.

$$\sum F_R = m\frac{v_T^2}{r}$$

$$mg - N = m\frac{v_T^2}{r}$$

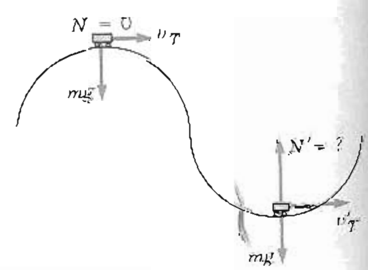


FIGURE 7-6 Example 7-4.

If the scale reads zero, $N = 0$ and

$$mg = m\frac{v_T^2}{r}$$

$$v_T = \sqrt{gr}$$

$$= \sqrt{9.8\ \text{m/sec}^2 \times 80\ \text{m}} = 28\ \text{m/sec}$$

In the dip $v_T = 40$ m/sec, mg is downward directed away from the center whereas N' now is upward toward the center. Following our sign convention we write

$$-mg + N' = m\frac{v_T'^2}{r}$$

$$N' = mg + m\frac{v_T'^2}{r}$$

$$= 600\text{ N} + \left(\frac{600\text{ N}}{9.8\text{ m/sec}^2}\right)\frac{(40\text{ m/sec}^2)}{(80\text{ m})}$$

$$= 600\text{ N} + 1224\text{ N} = 1824\text{ N}$$

Notice that now the scale reads more than three times the person's weight.

7.7 ORBITAL MOTION AND GRAVITATION

Johannes Kepler (1571–1630), a German astronomer and mathematician, plotted the orbits followed by the planets around the sun. He found three empirical rules for planetary motion, the first two were published in 1609 and the third in 1621. The reason for planetary behavior was not known until Newton found that he could derive Kepler's rules if he postulated a universal gravitational law. Namely, any two bodies are gravitationally attracted to each other by a force proportional to the product of their masses ($m_1 m_2$) and inversely proportional to the square of the distance between them, r^2. If we call the proportionality constant G, the *universal gravitational constant*, we may write

Johannes Kepler (1571–1630).

$$F = G\frac{m_1 m_2}{r^2} \qquad (7.21)$$

The value of this constant is $G = 6.67 \times 10^{-11}$ Nm^2/kg^2. Such a small number was not measurable in Newton's time, and it was first measured in 1798 by Henry Cavendish (1731–1810). Newton found a method of performing calculations without it. For example, he was able to calculate the acceleration of gravity at the earth's surface, $g = 9.8$ m/sec^2, which compared favorably with the experimental measurement of the acceleration of falling bodies. He reasoned as follows. Let m_e be the mass of the earth, m_o the mass of an object near the surface of the earth, m_m the mass of the moon, r_e the radius of the earth, and r_{em} the distance from the center of mass of the earth to the center of mass of the moon (assume constant radius of the moon's orbit).


Henry Cavendish (1731–1810)

At the surface of the earth the force on an object is its weight $m_o g$. This is equal to the force of gravity between the object and the earth, as given by Eq. 7.21. Considering all the mass of the earth to be at its center of mass, which is the geometrical center of a sphere, then

$$m_o g = G\frac{m_o m_e}{r^2} \qquad (7.22)$$

and

$$G = \frac{g r_e^2}{m_e} \tag{7.23}$$

The moon is also attracted to the earth by the gravitational force

$$F = G\frac{m_m m_e}{r_{em}^2} \tag{7.24}$$

This force is the centripetal force, which from Eq. 7.20 is

$$F_R = m_m \frac{v_T^2}{r_{em}}$$

Substituting Eq. 7.24 for this force yields

$$G\frac{m_m m_e}{r_{em}^2} = \frac{m_m v_T^2}{r_{em}}$$

from which

$$G = \frac{v_T^2 r_{em}}{m_e} \tag{7.25}$$

Equate the G's of Eqs. 7.23 and 7.25

$$g\frac{r_e^2}{\not{m}_e} = \frac{v_T^2 r_{em}}{\not{m}_e}$$

from which

$$g = \frac{v_T^2 r_{em}}{r_e^2} \tag{7.26}$$

He had the quantities $r_{em} = 3.8 \times 10^8$ m and $r_e = 6.3 \times 10^6$ m measured by astronomers. v_T is the speed of the moon, which is the distance around its orbit $2\pi r_{em}$ divided by the period of rotation of the moon around the earth, 27.3 days (2.36×10^6 sec)

$$v_T = \frac{2\pi \times 3.8 \times 10^8 \text{m}}{2.36 \times 10^6 \text{sec}} = 1.01 \times 10^3 \text{ m/sec}$$

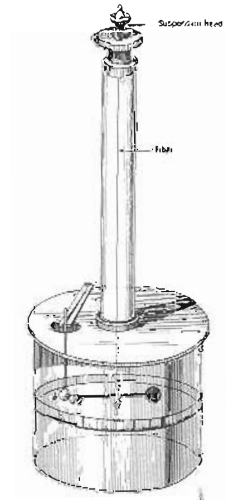
Substituting these numbers into Eq. 7.26 results in

$$g = \frac{(1.01 \times 10^3 \text{ m/sec})^2 (3.8 \times 10^8 \text{m})}{(6.3 \times 10^6 \text{m})^2}$$

$$= 9.8 \text{ m/sec}^2$$

And thus Newton was able to verify his gravitational law.

Later, when Cavendish measured G, the mass of the earth and the sun could be calculated (see Example 7-5).



Torsion balance used by Cavendish to determine the universal gravitational constant G.

EXAMPLE 7-5

The radius of the earth is $r_e = 6.3 \times 10^6$ m and $G = 6.67 \times 10^{-11}$ Nm^2/kg^2. Find the mass of the earth.

Solution Use Eq. 7.22

$$m_0 g = G\frac{m_0 m_e}{r_e^2}$$

$$m_e = \frac{r_e^2 g}{G}$$

$$= \frac{(6.3 \times 10^6 \text{m})^2 (9.8 \text{ m/sec}^2)}{6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2} = 5.8 \times 10^{24} \text{ kg}$$

The same answer can be obtained if we use Eq. 7.19 instead of Eq. 7.22.

$$F_R = mr\omega^2$$

Applying this to the motion of the moon around the earth, we write

$$G\frac{m_m m_e}{r_{em}^2} = m_m r_{em} \omega^2$$

The rotational velocity must be in rad/sec. One orbit of the moon is 2π *rad*, which it completes in 27.3 days. Therefore, $\omega = 2\pi/27.3$ day $= 2\pi/2.36 \times 10^6$ sec $= 2.66 \times 10^{-6}$ rad/sec. When this value is used the same answer is obtained for m_e.

$$m_e = \frac{r_{em}^3 \omega^2}{G}$$

$$= \frac{(3.8 \times 10^8 \text{m})^3 (2.66 \times 10^{-6} \text{ rad/sec})^2}{6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2}$$

$$= 5.8 \times 10^{24} \text{ kg}$$

PROBLEMS

7.1 A wheel of radius 0.5 m is rotating at 120 rev/min. (a) What is its rotational speed in rad/sec? (b) What is the tangential velocity of a point on the rim? (c) How many radians does the wheel turn in 10 sec? (d) If the wheel were rolling on the ground, what distance would it travel in 10 sec?

7.2 Calculate the angular velocity of the hour hand, the minute hand, and the second hand of a wristwatch.

7.3 A wheel rotating at 5 rev/sec coasts to rest in 30 sec. (a) What is its deceleration in rev/sec^2 and in rad/sec^2? (b) If the radius of the wheel is 0.4 m, what is the tangential acceleration of a point on the rim? (c) Through how many revolutions did the wheel turn in coming to rest?

Answer: (a) -0.167 rev/sec^2, -1.05 rad/sec^2, (b) -0.42 m/sec^2, (c) 75 rev.

7.4 A bicycle with a wheel radius of 0.34 m is traveling at 10 m/sec. What is the rotational speed of the wheels?

7.5 A wheel rotating at 10 rev/sec makes 1000 rev while coasting to a stop with constant deceleration. How long did it take to stop?

7.6 A wheel of radius 2 m starts rotating with constant angular acceleration $\alpha = 1.5\ \text{rad/sec}^2$. What are the tangential and radial accelerations of a point on the rim after the wheel has rotated 20π rad?

7.7 A pulley of radius $r_p = 8$ cm is connected to the shaft of an electric motor. A belt couples the pulley to a wheel of radius $r_w = 24$ cm (see Fig. 7-7). The motor shaft begins to rotate with an angular acceleration $\alpha = 25\ \text{rad/sec}^2$. (a) What is the angular velocity of the wheel after 3 sec? (b) Through what angle has the wheel rotated when the centripetal acceleration of a point on the rim of the wheel is $100g$? ($g = 9.8\ \text{m/sec}^2$)

Answer: (a) 25 rad/sec, (b) 245 rad.

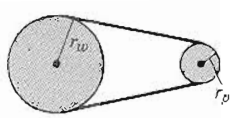


FIGURE 7-7 Problem 7.7.

7.8 A wheel makes 40 rev in 2 sec. The angular velocity at the end of the 2-sec time is 18 rev/sec. (a) What is the angular velocity at the beginning of the 2 sec? (b) What is the angular acceleration (assume it to be constant) of the wheel?

7.9 Assume that the orbit of the earth around the sun is circular and that the period of rotation is 365 days. The earth-sun distance is 1.5×10^{11} m. What is the centripetal acceleration of the earth resulting from its motion around the sun?

7.10 A 0.4-kg object on a string 0.5 m long attached to a pin on a frictionless table is made to rotate. If the breaking strength of the string is 20 N, what is the maximum rotational speed?

Answer: 10 rad/sec.

7.11 An object of mass 0.2 kg on a 0.4-m string is whirled in a vertical circle. (a) If the rotational speed is slowed until the object just completes the top of the circle with no tension in the string, what is its tangential velocity at that point? (b) If the same velocity is maintained at the bottom of the circle, what is the tension in the string at that point? See Fig. 7-8.

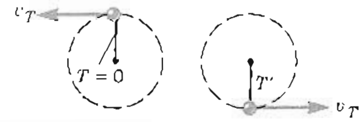


FIGURE 7-8 Problem 7.11.

7.12 A bug sits on a phonograph record 0.18 m from the center. If the record turns at 33 rev/min, what is the radial acceleration of the bug? If it has a mass of 0.5 gm, what is the centripetal force acting on it? See Fig. 7-9.

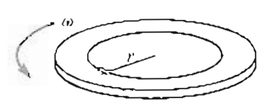


FIGURE 7-9 Problem 7.12.

7.13 The earth-sun distance is 1.5×10^{11} m. If the earth goes around the sun once in 365 days, find the mass of the sun. Assume the earth makes a circular orbit around the sun.

Answer: 2×10^{30} kg.

7.14 The force of attraction between oppositely charged particles is given by Coulomb's law, which has the same form as Newton's gravitational law

$$F = K\frac{q_1 q_2}{r^2}$$

where q_1 and q_2 are the charges on the particles in Coulombs (C), r is the distance between them, and K is a constant. In the Bohr model of the hydrogen atom, the electron revolves in a circular orbit around the stationary proton. The magnitude of the charge on the electron is the same as the charge on the proton $q_1 = q_2 = 1.6 \times 10^{-19}$ C and $K = 9 \times 10^9\ \text{Nm}^2/\text{C}^2$. The radius of the smallest electron orbit is 5.3×10^{-11} m, and the mass of the electron is 9.1×10^{-31} kg. Find the number of rev/sec of the electron around the proton, according to the model.

Answer: 6.56×10^{15} rev/sec.

7.15 What should the duration of a day be in order for a person standing at the equator to have the feeling of weightlessness, namely, for the normal force exerted by the ground on the person to be zero. The radius of the earth is 6.37×10^6 m.

Answer: 1.41 h.

7.16 (a) What is the centripetal acceleration of a person standing on the earth at a point of latitude 45°? (b) What is the magnitude and the direction of the force exerted by the ground on that person? Express your answer in terms of the weight mg of the person. The radius of the earth is 6.37×10^6 m.

7.17 A 2-kg block is rotating on a frictionless table with angular velocity $\omega = 2$ rev/sec. The block is connected to a 15-kg block by means of a string of total length 2 m that passes through a small hole in the table (see Fig. 7-10). How far below the tabletop does the 15-kg block hang?

Answer: 1.53 m.

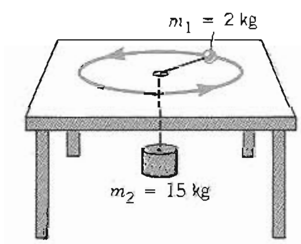


FIGURE 7-10 Problem 7.17.

7.18 A 1.5-kg mass is attached to one end of a rod of length $l = 1$ m and negligible weight. The other end of the rod is pivoted, and the mass rotates in a vertical circle. The tangential velocity of the mass at the top of the circle is 3 m/sec. (a) What is the magnitude and the direction of the force exerted by the rod on the mass at the top of the circle? (b) If friction at the pivot is negligible, what is the tangential velocity at the bottom of the circle? (c) What force does the rod exert on the mass at the bottom?

Answer: (a) 1.20 N upward, (b) 6.94 m/sec, (c) 87 N.

7.19 A particle of mass $m = 0.7$ kg is released from rest at point A in Fig. 7-11. It slides down and around the frictionless loop. (a) What are the radial and tangential accelerations at points B, C, and D? (b) What is the normal force exerted by the track at those three points? (This is the basis for a popular ride in amusement parks.)

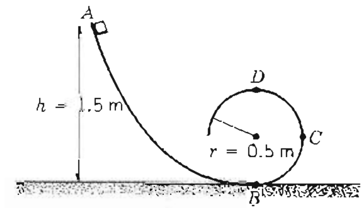


FIGURE 7-11 Problem 7.19.

7.20 The length of the string of a conical pendulum is 0.6 m (see Fig. 7-12). The mass of the bob is 1.2 kg. The angular velocity of the bob (which moves in a circle in the horizontal plane) is such that the angle between the string and the vertical is 30° and is constant. (a) What is the tension in the string? (b) What is the angular velocity of the bob?

Answer: (a) 13.6 N, (b) 4.34 rad/sec.

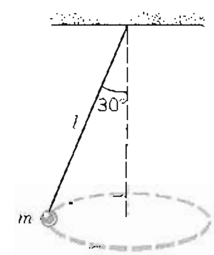


FIGURE 7-12
Problem 7.20.

7.21 A small particle of mass m is placed on top of a stationary, frictionless spherical ball of radius 0.5 m (see Fig. 7-13). It is given a slight kick to start sliding down. (a) Find the tangential velocity of the particle when it looses contact with the sphere. (b) What is the angle θ when contact is lost?

Answer: (a) 1.81 m/sec, (b) 48.2°.

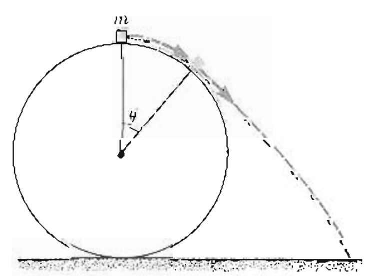


FIGURE 7-13 Problem 7.21.

8.1 INTRODUCTION

In this chapter we introduce the concepts of rotational dynamics. Previously, we developed the first principles of linear dynamics; now we adapt the principles of linear dynamics to rotating bodies. The same laws apply, but their formulation is different. We will see, however, that Newton's laws, momentum, energy, and power all have equations equivalent to their linear counterparts.

8.2 MOMENT OF INERTIA AND TORQUE

In Newton's second law, mass is the proportionality constant between force and acceleration. Newton called it the *inertial mass*, that is, the resistance of a body to having its state of motion changed. We encounter a similar concept in rotational motion. Independent of friction, it is easier to spin a bicycle wheel on its axle than it is to so spin a car wheel. This resistance to having the state of rotational motion changed is called the ***moment of inertia***, with symbol I. To demonstrate it in its simplest form, let us consider the rotation of a point mass m at one end of a rigid massless rod of length r. Let the other end of the rod be fastened to a point of rotation so that the system can rotate in the plane of the paper, as in Fig. 8-1. Suppose a force **F** is applied to the mass in the direction shown. We construct cartesian coordinates with the origin at m and x' as an extension of r. The force F has two components obtained by constructing the indicated lines perpendicular to the x' and y' coordinate axes. The x' component $F_R = F\cos\phi$ is in the direction of r. But because the rod is rigid, there can be no motion of the mass in the x' direction. The component in the y' direction is $F\sin\phi$. It should be observed that by construction this component is tangent to the circle of rotation at the point where m is located, so $F_T = F\sin\phi$. By Newton's second law, this tangential force causes a tangential acceleration

$$F_T = ma_T \tag{8.1}$$

From Eq. 7.7

$$a_T = r\alpha \tag{7.7}$$

Substituting for F_T and a_T in Eq. 8.1 yields

$$F\sin\phi = mr\alpha$$

Now multiply both sides of this equation by r

$$rF\sin\phi = mr^2\alpha \tag{8.2}$$

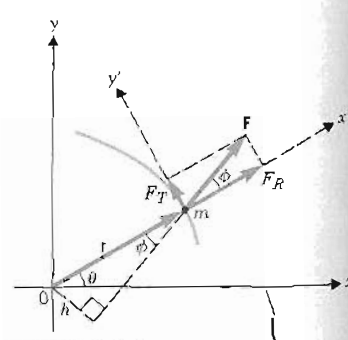


FIGURE 8-1

From Fig. 8-1, $r\sin\phi$ on the left side of Eq. 8.2 is equal to h, the perpendicular distance from the origin of the x-y coordinate system 0 to the line of F. Eq. 8.2 can therefore be rewritten as

$$Fh = mr^2\alpha \tag{8.3}$$

where

$$Fh = Fr \sin \phi$$

The quantity $Fh = Fr \sin \phi$, *the product of a force times the perpendicular distance from the point of rotation to the line of the force*, is called the *torque* produced by F, which is usually represented by τ, the small Greek letter *tau*. The quantity mr^2 on the right side of Eq. 8.2 is called the moment of inertia, I, of a point mass. We may therefore write Eq. 8.3 as

$$\tau = I\alpha \tag{8.4}$$

This is Newton's second law, which governs rotation. When compared with $\mathbf{F} = m\mathbf{a}$, we see that τ corresponds to force, I to mass, and α to linear acceleration. Just as in the linear case, τ must be the net torque and, if it is zero, there is no angular acceleration. Note that from the original definition, Eq. 7.6, the units of α are rad/sec^2.

It should be pointed out that if the force in Fig. 8-1 did not lie in the x-y plane but in some other plane while making the same angle with r, the torque would still have the same magnitude, $rF \sin \phi$, but the ensuing plane of rotation would not be the same. This ambiguity can be removed by assigning a direction to τ. It is conventional to define τ as the cross product of the position vector $\mathbf{r}$ and the force vector $\mathbf{F}$, namely,

$$\tau = \mathbf{r} \times \mathbf{F} \tag{8.5}$$

From the definition of the cross product, Eq. 2.2, the magnitude of τ is $rF \sin \phi$, which is the same value assigned previously. Moreover, the direction of τ is the perpendicular to the two vectors being crossed, $\mathbf{r}$ and $\mathbf{F}$, according to the right-hand rule discussed in Chapter 2. In the case illustrated by Fig. 8-1, τ is perpendicular to the plane of the paper outward. Equation 8.5 defines τ unambiguously.

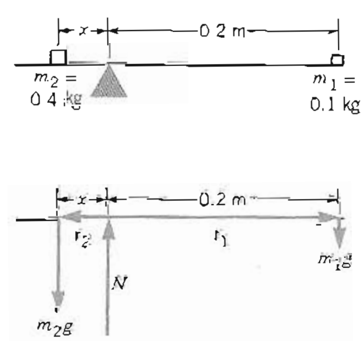


FIGURE 8-2 Example 8-1.

EXAMPLE 8-1

A balance scale consisting of a weightless rod has a mass of 0.1 kg on the right side 0.2 m from the pivot point. See Fig. 8-2. (a) How far from the pivot point on the left must 0.4 kg be placed so that a balance is achieved? (b) If the 0.4-kg mass is suddenly removed, what is the instantaneous rotational acceleration of the rod? (c) What is the instantaneous tangential acceleration of the 0.1-kg mass when the 0.4-kg mass is removed?

Solution

- (a) When a balance is achieved $\alpha = 0$ and therefore

$$\sum \tau = 0$$

On the right of the pivot point the force is $m_1 g$ downward and the cross product $\mathbf{r} \times \mathbf{F}$ is into the paper or negative. On the left the force is $m_2 g$ downward and the cross product $\mathbf{r} \times \mathbf{F}$ is out of the paper or positive.

$$(m_2 g)(x) \sin 90^\circ - (m_1 g)(0.2\,\text{m}) \sin 90^\circ = 0$$

Solving for x

$$x = \frac{(m_1)g(0.2\,\text{m}) \sin 90^\circ}{(m_2 g) \sin 90^\circ}$$

$$= \frac{(0.1\,\text{kg})(9.8\,\text{m/sec}^2)(0.2\,\text{m})}{(0.4\text{kg})(9.8\,\text{m/sec}^2)}$$

$$= 0.05\,\text{m}$$

- (b)

$$\alpha = \frac{\tau}{I} = \frac{(m_1 g)(0.2\,\text{m}) \sin 90^\circ}{(m_1)(0.2\,\text{m})^2}$$

$$\alpha = \frac{(0.1\,\text{kg})(9.8\,\text{m/sec}^2)(0.2\,\text{m}) \sin 90^\circ}{(0.1\,\text{kg})(0.2\,\text{m})^2}$$

$$\alpha = 49\,\text{rad/sec}^2 \text{ clockwise}$$

- (c)

$$a_T = r\alpha$$

$$= (0.2\,\text{m})(49\,\text{rad/sec}^2)$$

$$= 9.8\,\text{m/sec}^2$$

As expected, the answer to part (c) is the acceleration of a body in free fall. The same answer will be obtained for the left-hand weight if the right-hand one is removed except that the rod will rotate counterclockwise.

In Eq. 8.3, mr^2 is the moment of inertia of a point mass at a distance r from the pivot point. If there are a variety of masses at different distances from the pivot point, the moment of inertia of the assembly is the sum of their individual ones or

$$I = \sum_{i=1}^{n} m_i r_i^2 \qquad (8.6)$$

If all the masses are at the same distance r from the pivot point, $r_i^2 = r^2$ for all the terms in the sum and r^2 can be factored to obtain

$$I = r^2 \sum_{i=1}^{n} m_i \tag{8.7}$$

If we wish to find the moment of inertia of a thin hoop such as a bicycle wheel with essentially massless spokes, then the mass of the wheel M is simply

$$M = \sum_{i=1}^{n} m_i$$

where m_i is the mass of each infinitesimal element. Therefore, the moment of inertia of a bicycle wheel is approximately

$$I = Mr^2$$

The value of I for spheres, cylinders and such must be either derived by Eq. 8.6 or looked up in tables. We should note that unlike the translational inertia (the mass), the rotational inertia (moment of inertia) of an object depends on the location of the mass relative to the axis of rotation and in general is different for different axes of rotation (see problems 8.5 and 8.6).

8.3 ROTATIONAL KINETIC ENERGY

In Fig. 8-1 the force **F** was divided into two orthogonal components. It is seen that because r is fixed, the component of **F** in the x' direction can do no work.

In an infinitesimally small time interval dt, the tangential component of F, $F_T = F \sin \phi$ causes the particle to move an infinitesimal displacement ds, which from Eq. 7.2 is given by

$$ds = \mathbf{v}_T \, dt \tag{8.8}$$

Because time is a scalar quantity, the direction of ds is the same as that of $\mathbf{v}_T$, namely, tangent to the path of the particle and therefore in the same direction as $\mathbf{F}_T$ (see Fig. 8-3). The work done by $\mathbf{F}_T$ in this infinitesimal distance is dW, and by the definition of Eq. 5.3 is

$$dW = \mathbf{F}_T \cdot ds \tag{8.9}$$

But because $\mathbf{F}_T$ and ds are in the same direction, the dot product in Eq. 8.9 can be deleted.

$$dW = F_T \, ds \tag{8.10}$$

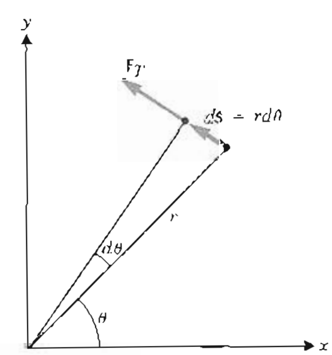


FIGURE 8-3

The tangential displacement of the particle is accompanied by an increase in the angle θ (see Fig. 8-3). The two are related by the differential form of Eq. 7.1

$$d\theta = \frac{ds}{r} \tag{7.1'}$$

Substituting $F \sin \phi$ for F_T and $r\,d\theta$ for ds into Eq. 8.10, we obtain

$$dW_\theta = F \sin \phi\, r\, d\theta \tag{8.11}$$

We have added the subscript θ to dW to indicate that the work is associated with an angular displacement $d\theta$.

The product $F \sin \phi\, r$ in the right side of Eq. 8.11 may be recognized as the torque exerted by the force F (see Eq. 8.5), therefore

$$dW_\theta = \tau\, d\theta \tag{8.12}$$

To find the work done for a finite rotation we simply integrate Eq. 8.12

$$W_\theta = \int_{\theta_0}^{\theta_f} \tau\, d\theta \tag{8.13}$$

where θ_0 and θ_f are the initial and final angles, respectively. Equation 8.13 is equivalent to the expression found in Chapter 5 for the work done in translation

$$W = \int_a^b F\, dx \tag{5.6}$$

In Chapter 5 we indicated that the net work done on a body changes its velocity, or more precisely, its kinetic energy. This was known as the work-energy theorem (Eq. 5.9). We can show that the same occurs in the case of rotation, while at the same time we will find an expression for the kinetic energy of a rotating body in terms of rotational parameters.

Substituting $\tau = I\alpha$ (Eq. 8.4) into Eq. 8.13 yields

$$W_\theta = \int_{\theta_0}^{\theta_f} I\alpha\, d\theta \tag{8.14}$$

But, by definition, $\alpha = \dfrac{d\omega}{dt}$ and $d\theta = \omega\, dt$; therefore,

$$W_\theta = \int_{\theta_0}^{\theta_f} I \frac{d\omega}{dt} \omega\, dt$$

The time dt cancels out in the integral, and we get

$$W_\theta = \int_{\omega_0}^{\omega_f} I\omega\, d\omega \tag{8.15}$$

where we have changed the limits from θ_0 and θ_f to ω_0 (initial angular velocity) and ω_f (final angular velocity), for we now integrate with respect to ω. If the moment of inertia is constant—that is, if the distance of the particle to the point of rotation does not change—then Eq. 8.15 becomes

$$W_\theta = I \int_{\omega_0}^{\omega_f} \omega \, d\omega$$

$$W_\theta = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_0^2 \tag{8.16}$$

Comparing this with the work-energy theorem of Chapter 5, where we saw that work done is equal to the change in kinetic energy, we may write

$$(\Delta E_k)_{rot} = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_0^2 \tag{8.17}$$

where the quantity $\frac{1}{2} I\omega^2$ is called the *rotational kinetic energy*, $(E_k)_{rot}$.

We may gain insight into the physical significance of $(E_k)_{rot} = 1/2I\omega^2$ by relating it to the linear kinetic energy of the particle. A point on a rotating system has an instantaneous tangential velocity v_T. Its kinetic energy is therefore

$$E_k = \frac{1}{2} m v_T^2$$

But, because $v_T = r\omega$, the kinetic energy may be written as

$$E_k = \frac{1}{2} m r^2 \omega^2$$

The moment of inertia of a point mass is $I = mr^2$. Therefore,

$$E_k = \frac{1}{2} I\omega^2$$

Hence, the expression for the rotational kinetic energy in terms of I and ω is simply another form of the kinetic energy of a particle rotating about a fixed axis.

The expression $\frac{1}{2} I\omega^2$ for the rotational kinetic energy can be readily shown to be applicable to the rotation of a rigid body made up of discrete masses m_i. The rotational kinetic energy of the ith particle is

$$(E_k)_{rot} \text{ of } i\text{th particle} = \frac{1}{2} m_i r_i^2 \omega_i^2 \tag{8.18}$$

and the $(E_k)_{rot}$ of the body is the sum of $(E_k)_{rot}$ of the individual masses

$$(E_k)_{rot} = \frac{1}{2} \sum_{i=1}^{n} m_i r_i^2 \omega_i^2$$

Because the body is rigid, all point masses rotate with the same angular velocity regardless of their distance from the axis, so $\omega_i^2 = \omega^2$ and it can be factored out of the sum. We then have

$$(E_k)_{rot} = \frac{1}{2} \omega^2 \sum_{i=1}^{n} m_i r_i^2$$

But the quantity in the summation is the definition of the moment of inertia I for a system of particles (see Eq. 8.6) and therefore

$$(E_k)_{\text{rot}} = \frac{1}{2} I\omega^2 \tag{8.19}$$

A body can be rotating as it translates through space; for example, the earth rotates about its axis as its center of mass moves about the sun. Clearly, the earth's rotation gives it more kinetic energy than if it were moving without rotation. Its total kinetic energy is therefore the sum of translational and rotational kinetic energies

$$(E_k)_{\text{total}} = (E_k)_{\text{trans}} + (E_k)_{\text{rot}} = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I\omega^2$$

where v_{CM} is the translational velocity of the center of mass.

EXAMPLE 8-2

A large wheel of radius 0.4 m and moment of inertia 1.2 kg-m^2, pivoted at the center, is free to rotate without friction. A rope is wound around it and a 2-kg weight is attached to the rope (see Fig. 8-4). When the weight has descended 1.5 m from its starting position (a) what is its downward velocity? (b) what is the rotational velocity of the wheel?

Solution

(a) We may solve this problem by the conservation of energy, equating the initial potential energy of the weight to its conversion to kinetic energy of the weight and of the wheel.

$$mgh = \frac{1}{2} m v^2 + \frac{1}{2} I\omega^2$$

The downward velocity v of the weight is equal to the tangential velocity at the rim of the wheel v_T; therefore

$$\omega = \frac{v_T}{r} = \frac{v}{r}$$

Substituting for ω

$$mgh = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{r^2}$$

We solve for the velocity v

$$v = \left[\frac{mgh}{\frac{1}{2} m + \frac{I}{2r^2}} \right]^{1/2}$$

$$= \left[\frac{(2\text{ kg})(9.8\text{ m/sec}^2)(1.5\text{ m})}{\left(\frac{1}{2}\right)(2\text{ kg}) + \frac{(1.2\text{ kg–m}^2)}{(2)(0.4\text{ m})^2}} \right]^{1/2}$$

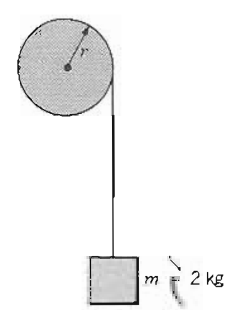


FIGURE 8-4 Example 8-2.

$$v = 2.5 \text{ m/sec}$$

(b) The answer to part (a) shows that any point on the rim of the wheel has a tangential velocity of $v_T = 2.5$ m/sec. We convert this to rotational velocity of the wheel

$$\omega = \frac{v_T}{r} = \frac{2.5 \text{ m/sec}}{0.4 \text{ m}} = 6.2 \text{ rad/sec}$$

8.4 POWER

The definition of *power* is work done per unit time. The incremental amount of work done in moving the mass in Fig. 8-3 a distance $ds = r\,d\theta$ is given in Eq. 8.12.

$$dW_\theta = \tau\,d\theta \tag{8.12}$$

But, from Eq. 5.15

$$\text{Power} = \frac{dW}{dt}$$

Substitute Eq. 8.12 to obtain

$$\text{Power} = \frac{\tau\,d\theta}{dt}$$

or, because $\omega = d\theta/dt$,

$$\text{Power} = \tau\omega \tag{8.20}$$

EXAMPLE 8-3

A machine shop has a lathe wheel of 40-cm diameter driven by a belt that goes around the rim. If the linear speed of the belt is 2 m/sec and the wheel requires a tangential force of 50 N to turn it, how much power is required to operate the lathe?

Solution Use Eq. 8.20

$$\text{Power} = \tau\omega$$

If there is no slipping between the belt and the wheel, the linear speed of the belt is equal to the tangential velocity at the rim of the wheel and the rotational velocity is therefore

$$\omega = \frac{v_T}{r}$$

$$= \frac{2 \text{ m/sec}}{0.2 \text{ m}}$$

$$= 10 \text{ rad/sec}$$

From Eq. 8.5 the torque is

$$\begin{aligned}\tau &= rF \sin\phi \\ &= (0.2\,\text{m})(50\,\text{N})\sin 90^\circ \\ &= 10\,\text{Nm}\end{aligned}$$

then

$$\begin{aligned}\text{Power} &= 10\,\text{Nm} \times 10\,\text{rad/sec} \\ &= 100\,\text{W}\left(\frac{1\,\text{hp}}{746\,\text{W}}\right) = 0.13\,\text{hp}\end{aligned}$$

8.5 ANGULAR MOMENTUM

We learned that an important property of a particle or of a system of particles (a body) is its momentum $\mathbf{p} = m\mathbf{v}$. An equivalent property can be associated with a rotating body.

Consider, as shown in Fig. 8-5, a particle of mass m with momentum $\mathbf{p} = m\mathbf{v}$ in the x-y plane. The position vector of m is $\mathbf{r}$, which is not required to be a constant.

From Newton's second law we write

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{4.2}$$

If we take the cross product of both sides with the position vector $\mathbf{r}$, we obtain

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) \tag{8.21}$$

By definition $\mathbf{r} \times \mathbf{F} = \tau$ and therefore

$$\tau = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) \tag{8.22}$$

The right side of Eq. 8.22 can be rewritten as

$$\mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) \tag{8.23}$$

The equivalence of the two expressions becomes evident if we differentiate the second term

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v})$$

But by definition $d\mathbf{r}/dt = \mathbf{v}$, the instantaneous velocity of the particle; therefore

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v})$$

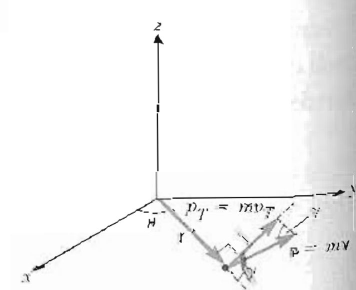


FIGURE 8-5

We saw in Chapter 2 that the cross product of two vectors in the same direction is zero. Therefore

$$\mathbf{v} \times m\mathbf{v} = m(\mathbf{v} \times \mathbf{v}) = 0$$

and, therefore

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v})$$

Substituting Eq. 8.23 for $\mathbf{r} \times \frac{d}{dt}(m\mathbf{v})$ in Eq. 8.22 yields

$$\tau = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) \tag{8.24}$$

We have already indicated that the torque τ in rotational motion plays the role of the force $\mathbf{F}$ in translational motion. Thus, if we compare Eq. 8.24 with Eq. 4.2, we are led to the conclusion that the quantity $\mathbf{r} \times m\mathbf{v}$ in rotational motion plays the same role as does the momentum $m\mathbf{v}$ in translation. We therefore call

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} \tag{8.25}$$

the *angular momentum* of the particle.

We can find another expression for $\mathbf{L}$ that shows even more clearly its correspondence to the momentum $m\mathbf{v}$.

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = rmv \sin \gamma$$

where γ is the angle between the radius vector $\mathbf{r}$ and the linear momentum $m\mathbf{v}$ (see Fig. 8-5). But, from Fig. 8-5, $mv \sin \gamma = mv_T$, and

$$L = rmv_T$$

From Eq. 7.4, $v_T = r\omega$ and therefore

$$L + mr^2\omega \tag{8.26}$$

We have defined mr^2 as the moment of inertia I of a point mass; hence

$$L = I\omega \tag{8.27}$$

The equation of motion for rotation, Eq. 8.24, can be written

$$\tau = \frac{dL}{dt} \quad \text{or} \quad \tau = \frac{d(I\omega)}{dt} \tag{8.28}$$

8.6 CONSERVATION OF ANGULAR MOMENTUM

We will now show that the law of conservation of momentum that was derived in Chapter 6 applies equally to angular momentum. Start with Eq. 8.28

$$\tau = \frac{d(I\omega)}{dt} = \frac{dL}{dt}$$

If we have a situation in which there is no net externally applied torque, then $\tau = 0$. Thus

$$\frac{dL}{dt} = 0$$

and $L =$ constant. Hence, $I\omega =$ constant.

Therefore, with no net external torque

$$(I\omega)_0 = (I\omega)_f \qquad (8.29)$$

This is known as the law of *conservation of angular momentum*.

EXAMPLE 8-4

Suppose the body of an ice skater has a moment of inertia $I = 4\,\text{kg-m}^2$ and her arms have a mass of 5 kg each with the center of mass at 0.4 m from her body. She starts to turn at 0.5 rev/sec on the point of her skate with her arms outstretched. She then pulls her arms inward so that their center of mass is at the axis of her body, $r = 0$. What will be her speed of rotation?

Solution

$$I_0\omega_0 = I_f\omega_f$$

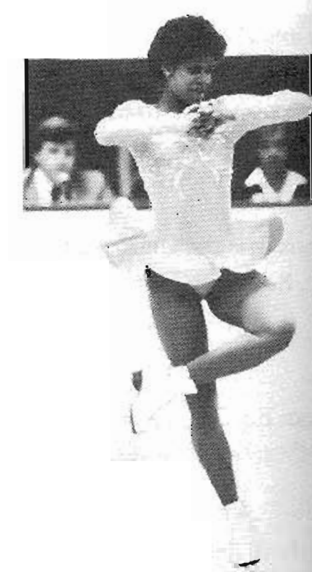
$$(I_{\text{body}} + I_{\text{arms}}\omega_0 = I_{\text{body}}\,\omega_f$$

$$(I_{\text{body}} + 2\,mr^2)\omega_0 = I_{\text{body}}\,\omega_f$$

Solving for ω_f

$$\omega_f = \frac{(I_{\text{body}} + 2\,mr^2)\,\omega_0}{I_{\text{body}}} = \frac{[4\,\text{kg-m}^2 + 2 \times 5\,\text{kg} \times (0.4\,\text{m})^2](0.5\,\text{rev/sec})}{4\,\text{kg-m}^2}$$

$$= 0.7\,\text{rev/sec}$$



Due to the small torque exerted by the ice, the angular momentum of a spinning skater is almost constant. As a result, when the skater pulls her arms inward, thus reducing her moment of inertia, her angular velocity increases.

PROBLEMS

8.1 A bicycle wheel of mass 2 kg and radius 0.32 m is spinning freely on its axle at 2 rev/sec. When you place your hand against the tire the wheel decelerates uniformly and comes to a stop in 8 sec. What was the torque of your hand against the wheel?

8.2 Two masses, $m_1 = 1$ kg and $m_2 = 5$ kg, are connected by a rigid rod of negligible weight (see Fig. 8-6). The system is pivoted about point 0. The gravitational forces act in the negative z direction. (a) Express the position vectors and the forces on the masses in terms of unit vectors and calculate the torque on the system. (b) What is the angular acceleration of the system at the instant shown in Fig. 8-6?

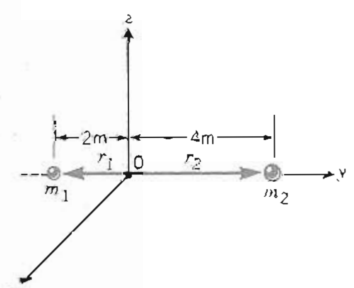


FIGURE 8-6 Problem 8.2.

8.3 A roulette wheel with $I = 0.5$ kg-m^2 rotating initially at 2 rev/sec coasts to a stop from the constant friction torque of the bearing. If the torque is 0.4 N-m, how long does it take to stop?

Answer: 15.7 sec.

8.4 A grindstone with $I = 240$ kg-m^2 rotates with a speed of 1 rev/sec. A knife blade is pressed against it, and the wheel coasts to a stop with constant deceleration in 12 sec. What torque did the knife exert on the wheel?

8.5 Four identical masses ($m = 2$ kg) are connected by rods of negligible weight to form a rectangle (see Fig. 8-7). The masses are rotated about an axis perpendicular to the plane of the rectangle and passing through its center with an angular acceleration $\alpha = 3$ rev/sec^2. What torque is needed?

Answer: 1282 N-m.

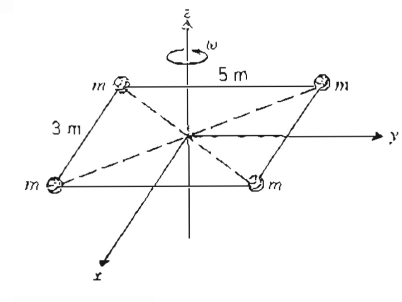


FIGURE 8-7 Problem 8.5.

8.6 Repeat problem 8.5 for a rotation, with the same angular acceleration, about an axis through a corner of the rectangle (see Fig. 8-8).

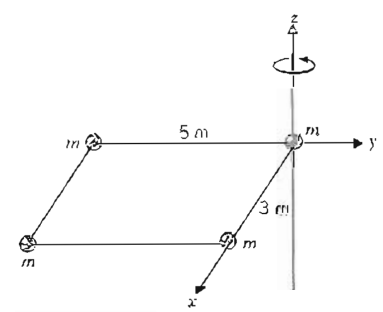


FIGURE 8-8 Problem 8.6.

8.7 A uniform wooden board of mass 20 kg rests on two supports as shown in Fig. 8-9. A 30-kg steel block is placed to the right of support A. How far to the right of A can the steel block be placed without tipping the board?

Answer: 2.0 m.

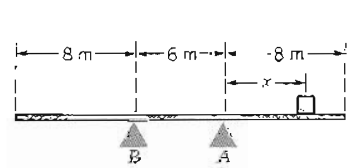


FIGURE 8-9 Problem 8.7.

8.8 A wheel ($I = 30$ kg-m^2) is pivoted about an axis through its center. A 90 N-m torque is applied to the wheel, which then accelerates from rest to an angular velocity of 20 rad/sec in 10 sec. (a) What is the friction torque of the bearings? (b) If the applied torque is removed after 60 sec, how long will it take for the wheel to come to rest?

8.9 Calculate the change in rotational kinetic energy of the roulette wheel and grindstone of problems 8-3 and 8-4.

8.10 A ball of mass 0.3 kg and radius 0.1 m rolls along the ground with a transverse speed of 4 m/sec. It comes to a slope inclined at 30°. How far up the slope does it roll? (I (ball) $= 2/5\, mr^2$).

Answer: 2.29 m.

8.11 A wheel of moment of inertia 60 kg-m^2 and radius 1.5 m is rotating about an axis through its center with an angular velocity $\omega = 30$ rev/sec. A brake is applied producing a normal force of 450 N against the rim (see Fig. 8-10). The coefficient of friction between the wheel and the brake is 0.5. (a) How long will it take for the wheel to stop? (b) Calculate the work done by friction, and show that this is equal to the change in the kinetic energy of the wheel.

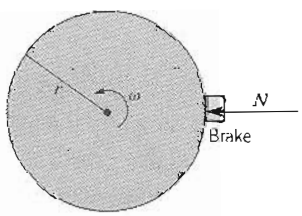


FIGURE 8-10 Problem 8.11.

8.12 Although most people are concerned about the horsepower of a car's motor, the important parameter is the amount of torque that can be given to the rear wheels. The torque of the motor is turned at right angles to the wheels by the differential gear. Assume that in low gear the angular velocity of the rear wheels is 0.1 that of the motor. If the motor has 200 hp and is turning over at a rate of 1400 rev/min, how much torque is delivered to the rear wheels?

Answer: 1.02×10^4 N-m.

8.13 A 4-kg block is attached to one end of a light rope. The other end of the rope is wrapped around a pulley of moment of inertia $I = 0.5$ kg-m^2 and radius $r = 0.2$ m (see Fig. 8-11). The block is released from rest, and it moves down 9 m in 3 sec. (a) What is the friction torque of the bearings? (b) Use energy principles to calculate the velocity of the block after it has fallen 9 m.

Answer: (a) 1.24 N-m, (b) 6.0 m/sec.

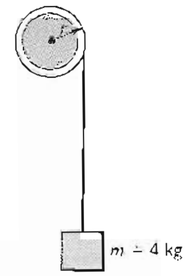


FIGURE 8-11
Problem 8.13.

8.14 A 2-kg block resting on a frictionless table is connected by a string passing over a pulley to a second block, $m_2 = 5$ kg, hanging over the edge of the table 0.8 m above the floor (see Fig. 8-12). The moment of inertia of the pulley is 0.8 kg-m^2 and the radius is 0.1 m. Neglect the friction of the bearings and assume that there is no slipping between the string and the pulley. Use energy methods to calculate the velocity of m_2 as it hits the floor.

Answer: 0.95 m/sec.

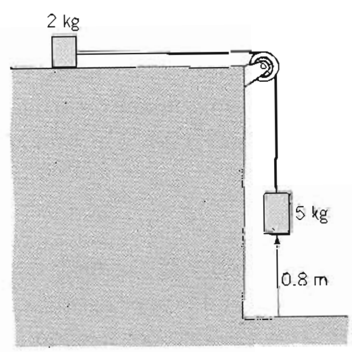


FIGURE 8-12 Problem 8.14.

8.15 Repeat problem 8.14 if the coefficient of friction between the 2-kg block and the table is 0.25.

8.16 Use the data in problem 7.14 to find the angular momentum of an electron in the smallest orbit of the hydrogen atom.

Answer: 1.05×10^{-34} J-sec.

8.17 A 50-gm mouse falls onto the outer edge of a phonograph turntable of radius 20 cm rotating at 33 rev/min. How much work must it do to walk into the center post? Assume that the angular velocity of the turntable does not change.

8.18 A children's merry-go-round of radius 4 m and mass 100 kg has an 80-kg man standing at the rim. The merry-go-round coasts on a frictionless bearing at 0.2 rev/sec. The man walks inward 2 m toward the center. What is the new rotational speed of the merry-go-round? What is the source of this energy? (The moment of inertia of a solid disk is $I = 1/2\, mr^2$).

Answer: 0.37 rev/sec.

8.19 A mass of 0.1 kg on a string is rotating on a frictionless table with $\omega = 1$ rev/sec and $r = 0.2$ m. The string passes through a hole in the table and is held by a hand below (see Fig. 8-13). (a) What is the angular momentum of the mass? (b) What is the kinetic energy of the mass? (c) If the string is pulled down by the hand until $r = 0.1$ m, what is the new rotational speed of the mass? (d) What is the new kinetic energy? (e) How much work did the hand do in pulling the string?

Answer: (a) 2.51×10^{-2} J-sec, (b) 7.9×10^{-2} J, (c) 4 rev/sec, (d) 3.16×10^{-1} J, (e) 2.37×10^{-1} J

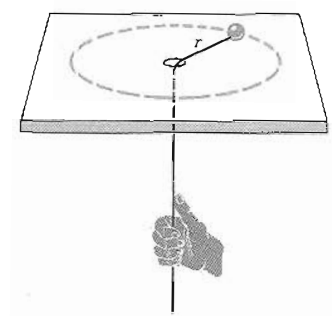


FIGURE 8-13 Problem 8.19.

9.1 INTRODUCTION

Heat and temperature proved to be very elusive concepts to early scientists. Most historic theories—for examples, the phlogiston and the caloric—assumed that heat was a substance that could flow, much as a gas or a fluid. In fact, the mathematics of heat flow were correctly worked out before scientists learned the true nature of heat and its associated property, temperature. For indeed "heat" does flow; but what is heat?

Our modern understanding of heat, temperature, and the behavior of gases is the result of two and a half centuries of scientific investigation; we now know that heat is a form of energy. We will not tell the entire story, because it is beyond the scope of this book. We will, however, trace the story through the measurement of temperature, the ideal gas law, and then the application of the first principles of mechanics, as developed in the earlier chapters, to the average motion of molecules in gases. The identification of the average kinetic energy of molecules with temperature will then be shown. From that we will be able to write the first law of thermodynamics, which is a broadened statement of the law of conservation of energy.

9.2 MOLECULAR WEIGHT

In real systems there are vast numbers of atoms, all of which obey the first principles of physics, or quantum variations of them, in their motion. The motion of each is different however, so a conclusion about the behavior of a group of atoms is statistical. In this book we will consider only systems composed of identical atoms, or molecules such as oxygen molecules (O_2) or nitrogen molecules (N_2).

Ensembles of different atoms or molecules have different statistical averages of their properties. This is because if objects that have the same kinetic energies, $\frac{1}{2}mv^2$, have different masses, then their velocities are different. It is therefore important to know the mass of the atoms or molecules that make up the ensemble. It is clear that if we know the mass of the ensemble and the weight of each particle of the ensemble, then we can immediately determine the number of particles.

Amedo Avogadro (1776–1856).

We use a unit called the ***mole*** (abbr. ***mol***) as a measure of the number of particles with the following definition. *A mole of a substance is that quantity which contains the same number of particles as there are atoms in 12 g* (12×10^{-3} *kg*) *of carbon-12*. The measure in SI units is the ***kilomole (kmol)***, which is the quantity of the substance that contains the same number of particles as there are atoms in 12 kg of carbon-12. All the elements have *isotopes*, that is, atoms with the same chemical properties but with a slightly different mass. Therefore, a single isotope of carbon (carbon-12) is chosen as the reference standard. This standard is said to have a mass of exactly 12 u per atom, where u is called an *atomic mass unit* and has the value

$$1\text{ u} = 1.66057 \times 10^{-27}\text{ kg}$$

The mass of an atom (or molecule) in atomic mass units is called the *atomic weight* (or *molecular weight*). Thus, for example, the atomic weight of carbon-12 is 12 u, that of hydrogen-1 is 1.0078 u. The mass, in grams, of a mole of a substance is *numerically* equal to the atomic weight (or molecular weight) of the atoms of that substance, and it is referred to as the *gram atomic weight* (or *gram molecular weight*). The mass of 1 mole of carbon-12 is 12 g/mole, that of hydrogen-1 is 1.0078 g/mole. Often, the word "gram" is deleted from the expressions for the mass of a mole, which may lead to confusion. We can rely on the units to see whether we are dealing with the mass of an atom or that of a mole.

By the use of very careful techniques, chemists and physicists have been able to measure the number of atoms in 1 mole of a substance. This is called *Avogadro's number* and has the value

$$N_A = 6.022 \times 10^{23}\text{ atoms/mol} = 6.022 \times 10^{26}\text{ atoms/kmol}$$

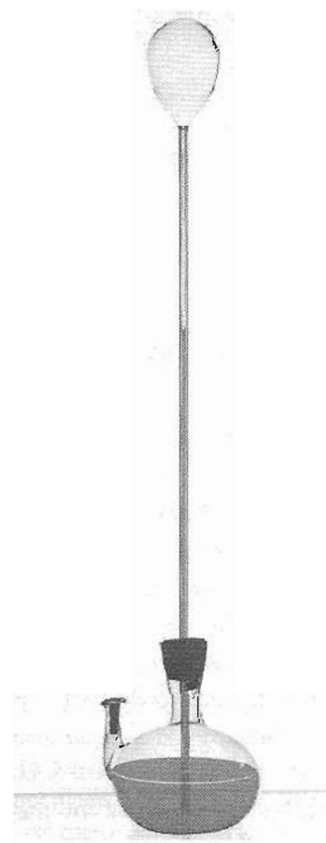
Therefore, if we know the number of moles of substance present in the ensemble, we can calculate the number of atoms or molecules present.

We will use the symbol n to represent the number of moles present, where n may be greater or less than one, or equal to it. Because one usually has less than 1 mole, n has the name *mole fraction*. If we denote M as the gram molecular weight of a substance and m as the mass of the amount present, then $n = m/M$ and, because M has N_A atoms or molecules, the number of atoms or molecules present is nN_A.

9.3 THERMOMETERS

It has been known from ancient times that solids and liquids expand when they are heated. It is not known when the first thermometers were made, but they are believed to have been brought into general use by Duke Ferdinand of Tuscany in 1654. They were generally used shortly thereafter by members of the Academy of Science of Florence (which was founded by him) and were long known as Florentine thermometers. They were much like modern thermometers in that they had a colored liquid, presumably alcohol, hermetically sealed in a tube with a bulb at one end, with little pieces of colored glass to mark even divisions on the scale.

In 1714 Gabriel Fahrenheit proposed that a scale be established in which the temperature of the human body be taken as 100° (which has since been corrected to 98.6°) and 0° be the lowest temperature attainable with a mixture of ice and salt, sodium chloride (NaCl). Using this scale, the melting point of pure ice is 32° and the boiling point of pure water at sea level is 212°. Shortly after Fahrenheit's death in



One of the earliest thermometers used was Galileo's thermoscope shown here.

1736 a different scale, Centigrade or Celsius, came into use; by this scale, the melting point of ice was taken at 0°C and the boiling point of water at 100°C.

The Fahrenheit scale is still popularly used in the United States and Canada, probably because of its finer divisions for meteorologic measurements, but in scientific laboratories and in most of the world the Celsius scale is used. The conversion between the two has ever since been confusing to the layperson, but, with a little thought, one can eliminate the difficulty. The conversion is easy to see if we construct yet a third scale, which we will call the °F-32 scale, in Fig. 9-1. If we take an arbitrary temperature point on the °C scale and the same point on the °F-32 scale, we may make a ratio between the temperature of °C and °F-32 scales as

$$\frac{°\mathrm{C}}{°\mathrm{F} - 32} = \frac{100}{180}$$

or

$$\frac{°\mathrm{C}}{°\mathrm{F} - 32} = \frac{5}{9}$$

Thus

$$°\mathrm{C} = \frac{5}{9}(°\mathrm{F} - 32) \qquad (9.1)$$

or

$$°\mathrm{F} = \frac{9}{5}°\mathrm{C} + 32 \qquad (9.2)$$

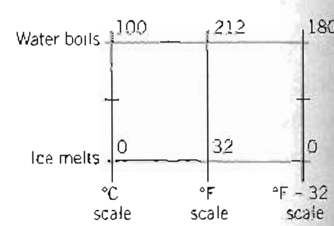


FIGURE 9-1

9.4 IDEAL GAS LAW AND ABSOLUTE TEMPERATURE

We define the term *pressure*, P, as force perpendicular to a surface per unit surface area, or $P = F/A$. Therefore, for a given force, the smaller the area on which it acts the larger is the pressure. For example, a weight terminating in a sharp point can usually make an indentation in a surface on which the point rests. But if the point is changed to a flat face of larger area, no mark will be made.

The dimension of pressure is newton/meter^2 (N/m^2), and in fluids we sometimes use the term pascal (Pa) where 1 Pa = 1 N/m^2. Atmospheric pressure at sea level is approximately 1.01×10^5 N/m^2, which is equivalent in the English system to 14.7 lb/in^2.

An ideal gas is one that has no tendency to condense. This means that the atoms are infinitesimal in size and that there is no attractive force between them. Ideal gases do not exist, but they may be approximated by rare gases (such as helium, neon, argon) at low pressure, or any other gas at *very* low pressure. Robert Boyle in 1662 showed that if the quantity of gas and its temperature remain constant, then the pressure and


Robert Boyle (1627–1691).

volume vary inversely.

$$P = \frac{C}{V} \tag{9.3}$$

where C is a constant. In 1802, Joseph Gay-Lussac showed that if the quantity of gas and its volume remained constant, the pressure is proportional to the temperature or

$$P = KT \tag{9.4}$$

where K is a constant.

These two laws may be combined into a single law with a single constant:

$$PV = R'T \tag{9.5}$$

When equal volumes of the same gas are taken at the same temperature and pressure, R' remains constant. If, however, equal masses of different gases are taken under the same conditions, R' varies inversely with the molecular weight.

In the middle of the last century low temperatures were achieved in Lord Kelvin's laboratory in England. The ideal gas law, Eq. 9.5, was examined over an extended range of temperatures. It seemed desirable to establish R' as a constant and to correct its inverse variation with the mass of the gas used by multiplying by the number of moles, n (because $n = m/M$). With the introduction of this term n the ideal gas law is written as

$$PV = nRT \tag{9.6}$$

where n is the number of moles (mole fraction) and R is now the same for all gases.

When Kelvin's group examined Eq. 9.6 at constant volume for different amounts of a gas $n_1, n_2, n_3, \ldots$, the data appeared as in Fig. 9-2. It is seen in this graph that the data taken to the lowest achievable temperature T_L all lie on straight lines and, if these lines are extrapolated to $P = 0$, they terminate at a common point, -273.16° C. This was called ***absolute zero***, and is the lowest possible temperature. It was therefore logical to establish a new temperature scale with its zero point at -273.16° C. Thus, $0^\circ\text{C} = +273.16$ K, where K is the symbol for the new scale, called the Kelvin or ***absolute*** scale. It is related to the Celsius scale as

$$\text{K} = 273.16^\circ + {}^\circ\text{C} \tag{9.7}$$

With data of the type shown in Fig. 9-2, the gas constant R was evaluated by Kelvin and his associates as

$$R = 8314\ \text{J/kmol} - \text{K}$$

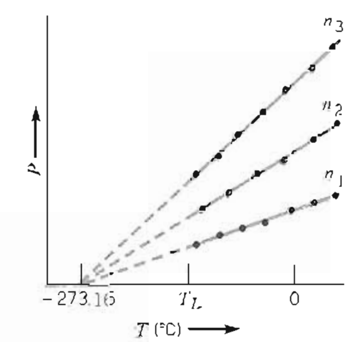


FIGURE 9-2. Results of Lord Kelvin's experiments showing a linear variation of the pressure of a gas with temperature when the volume is kept constant. The three curves correspond to different amounts of gas. When the curves are extrapolated, they terminate at a common point $T = -273.16^\circ$.

EXAMPLE 9-1

What is the temperature of absolute zero on the Fahrenheit scale?

Solution Absolute zero = −273.16°C. From Eq. 9.2

$$°F = \frac{9}{5}°C + 32$$

$$°F = \frac{9}{5} \times (-273.16) + 32 = -459.7°F$$

EXAMPLE 9-2

In a typical experiment to determine the value of the gas constant R, 0.152 g of neon gas (atomic weight 20.2 g/mole) is introduced into a 100-cm³ flask that is closed and attached to a pressure gauge. It is found that when the flask is placed in a constant temperature bath at 50°C the pressure of the gas is 2 atmospheres (atm). What value of R is obtained?

Solution The mole fraction n is the ratio of the number of grams present to the atomic weight in grams.

$$n = \frac{0.152\,\text{g}}{20.2\,\text{g/mole}} = 7.52 \times 10^{-3}\,\text{mol} = 7.52 \times 10^{-6}\,\text{kmol}$$

The volume is

$$V = 10^2\,\text{cm}^3 \left(\frac{1\,\text{m}}{10^2\,\text{cm}}\right)^3 = 10^{-4}\,\text{m}^3$$

The pressure is

$$P = 2\,\text{atm}\left(\frac{1.01 \times 10^5\,N/m^2}{1\,\text{atm}}\right) = 2.02 \times 10^5\,\text{N/m}^2$$

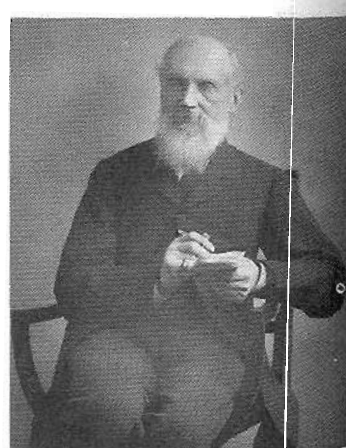
The temperature in K is $T = 273° + 50° = 323$ K.

The ideal gas law $PV = nRT$ is written as

$$R = \frac{PV}{nT}$$

$$= \frac{2.02 \times 10^5\,\text{N/m}^2 \times 10^{-4}\,\text{m}^3}{7.52 \times 10^{-6}\,\text{kmol} \times 323\,\text{K}}$$

$$= 8316\,\text{J/kmol} - \text{K}$$



William Thompson Kelvin (1824–1907).

EXAMPLE 9-3

In a diesel engine, no spark plug is required because the temperature is raised to the ignition point of the air-fuel mixture by compression. In a typical diesel engine the air intake is at 27°C and at a pressure of 1 atm, and it is compressed to 1/15 of its original volume with its pressure becoming about 50 atm. What is the temperature of the air-fuel mixture in the cylinder in °C?

Solution We note from the ideal gas law that

$$\frac{PV}{T} = nR$$

and, if the quantity of gas is kept constant, then the right side of the equation is a constant. Therefore, if we change any of the quantities on the left side, the other quantities must change to yield the same constant, nR. This means that the initial conditions of the left side must equal the final conditions because both the initial and the final conditions are equal to the same constant, nR. We write this as

$$\frac{P_0 V_0}{T_0} = \frac{P_f V_f}{T_f}$$

where T must be in K.

In this problem

$$\begin{aligned} T_f &= \frac{P_f}{P_0}\frac{V_f}{V_0}T_0 \\ &= \frac{50\,\text{atm}}{1\,\text{atm}}\frac{V_0/15\,\text{m}^3}{V_0\text{m}^3} \times 300\,\text{K} \\ &= 1000\,\text{K} = 727°\,\text{C} \end{aligned}$$

9.5 KINETIC THEORY OF GAS PRESSURE

We will now show how the concept of momentum conservation and the definition of pressure can be used to calculate the statistical behavior of a large number of atoms or molecules in a gas. One of the assumptions in this calculation is that all collisions between atoms or molecules are perfectly elastic. This is not strictly true at high temperatures, because in some high-energy collisions electrons are excited or even knocked off atoms. Although this situation can be dealt with theoretically, it will not concern us here.

Because the walls of a container are also made of atoms, then all collisions between the atoms of a gas in a container and the walls are elastic. One other fact must be kept in mind. In an elastic collision of an atom with the container wall, the velocity component normal to the wall is reversed on collision with its magnitude unchanged, and the velocity component in the direction parallel to the surface of the wall is unchanged. This can be seen in the two-dimensional schematic of Fig. 9-3. Viewed from above, it is clear that a v_z velocity component would also be unchanged.

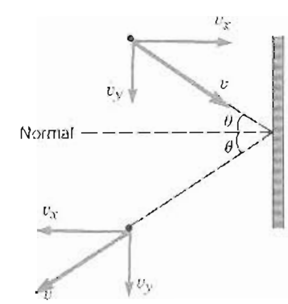


FIGURE 9-3 Two-dimensional representation of an elastic collision of a molecule with the wall of the container.

We recall from Chapter 4 that an impulse acting on a body is equal to the change in the momentum of the body. For the situation of Fig. 9-3,

$$\overline{F}_x\,\Delta t = \Delta\,(mv_x)$$

where $\overline{F}_x$ is the average force exerted by the wall of the container on the atom during the time interval Δt, and where m is the mass of the atom. From Newton's third law of action and reaction, the magnitude of the force exerted by the atom on the wall is equal to $\overline{F}_x$ and can be written as

$$\overline{F}_x = \frac{mv_{x\text{ final}} - mv_{x\text{ initial}}}{\Delta t}$$

Because only the direction and not the magnitude of v_x changes on collision

$$\overline{F}_x = \pm\frac{2mv_x}{\Delta t} \tag{9.8}$$

where the choice of sign depends on the assignment of velocity direction sign.

Suppose the atom is moving about in a cubical box of side length l, area of a face $A = l^2$ and a volume $V = l^3$ (see Fig. 9-4). The direction of the force in the impulse of Eq. 9.8 and the initial velocity are reversed if the atom collides with the opposite wall. Thus, it will be convenient to consider only the magnitude of the average force, so we will drop the negative sign in Eq. 9.8. We may approximate Δt of Eq. 9.8 as the time between collisions of the atom against the wall. This is the time for the atom to travel to the opposite wall, bounce off it, and return to the first wall.

Because the atom's velocity in the x direction remains constant in magnitude, the time for a round trip between opposite walls is

$$\Delta t = \frac{2l}{v_x} \tag{9.9}$$

Substituting this into Eq. 9.8 obtains the magnitude of the average force on a wall due to the successive striking by one atom

$$\overline{F}_x = \frac{2mv_x}{\frac{2l}{v_x}} = \frac{mv_x^2}{l} \tag{9.10}$$

This is the average force on a wall in the y-z plane due to a single atom of the gas. Let us call the force due to this atom $\overline{F}_{x1}$ and the x velocity v_{x1}. Then if there is a second atom with velocity v_{x2}, it would contribute a force $\overline{F}_{x2}$, and so on. The total average force on a wall due to the x motion of N atoms in the box would be the sum of the contribution of each, or

$$\begin{aligned}\overline{F}_x &= \frac{m}{l}v_{x1}^2 + \frac{m}{l}v_{x2}^2 + \cdots \frac{m}{l}v_{xN}^2 \\ &= \frac{m}{l}(v_{x1}^2 + v_{x2}^2 + \cdots v_{xN}^2)\end{aligned} \tag{9.11}$$

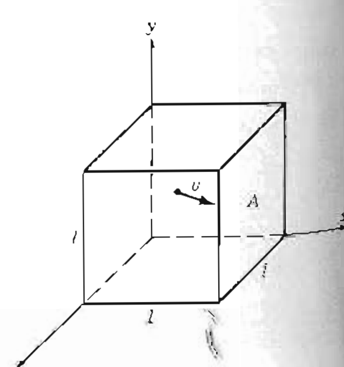


FIGURE 9-4

If N is the total number of atoms in the box, then by the definition of an average as the sum of the individual amounts divided by the number of items, we may write for $\overline{v_x^2}$, the average of the squared individual x velocities,

$$\overline{v_x^2} = \frac{v_{x1}^2 + v_{x2}^2 + \cdots v_{xN}^2}{N} \tag{9.12}$$

Substitute Eq. 9.12 into Eq. 9.11

$$\overline{F}_x = \frac{mN}{l}\overline{v_x^2} \tag{9.13}$$

By the three-dimensional pythagorean theorem

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

or, expressed in averages,

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} \tag{9.14}$$

But in a gas in equilibrium there is no preferred direction of motion; hence

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

Therefore Eq. 9.14 may be written as

$$\overline{v^2} = \overline{3v_x^2} \tag{9.15}$$

Substitute $\overline{v_x^2}$ from Eq. 9.15 into Eq. 9.13, and $\overline{F}_x$ becomes a general force on any wall $\overline{F}$

$$\overline{F} = \frac{mN}{3l}\overline{v^2} \tag{9.16}$$

If we now use the definition of pressure $P = F/A$, we may write Eq. 9.16 as

$$P = \frac{mN}{3Al}\overline{v^2}$$

$$P = \frac{mN}{3V}\overline{v^2} \tag{9.17}$$

where $V = Al$ is the volume of the box. Now multiply and divide Eq. 9.17 by 2 and obtain

$$P = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}m\overline{v^2}\right) \tag{9.18}$$

which shows that the pressure of a gas on the walls of a container is proportional to the average kinetic energy of the atoms or molecules of the gas. One should recognize that atom-atom collisions also take place in a gas. In a more complete calculation these are considered, but the result given by Eq. 9.18 remains unchanged.

9.6 KINETIC THEORY OF TEMPERATURE

We may now show the relation of molecular motion to temperature by using the ideal gas law, Eq. 9.6

$$PV = nRT \tag{9.6}$$

and the equation we just derived for the pressure, Eq. 9.18. Substituting for the pressure in Eq. 9.6 from Eq. 9.18, we obtain

$$\frac{2}{3}N\left(\frac{1}{2}m\overline{v^2}\right) = nRT \tag{9.19}$$

in which it should be recalled that N is the number of molecules present in the box and n is the number or fraction of moles present. By definition

$$n(\text{number of moles}) = \frac{N(\text{number of molecules})}{N_A(\text{Avogadro's number})}$$

Substituting $n = N/N_A$ in Eq. 9.19 gives

$$\frac{2}{3}N\left(\frac{1}{2}m\overline{v^2}\right) = \frac{N}{N_A}RT \tag{9.20}$$

The number of molecules, N, cancels, and we are left with the ratio of two constants R/N_A. This ratio occurs so frequently that it is given the name *Boltzmann's constant*, after the German theorist, with the symbol k_B. Its value is

$$\begin{aligned} k_B &= \frac{R}{N_A} \\ &= \frac{8314\ \text{J/kmol}\cdot\text{K}}{6.02\times10^{26}\ \text{molecule/kmol}} \\ &= 1.38\times10^{-23}\ \text{J/K per molecule} \end{aligned}$$

Eq. 9.20 then becomes

$$\frac{1}{2}m\,\overline{v^2} = \frac{3}{2}k_B\,T \tag{9.21}$$

or

$$T = \frac{2}{3}\frac{\overline{E_k}}{k_B} \tag{9.22}$$

and we have shown that temperature is simply proportional to the average kinetic energy E_k of the molecules. Although this calculation has been done for gases, the same result is obtained for liquids and solids.

We may use Eq. 9.21 to find the speed of molecules in a gas. Note, however, that the speed will actually be the square root of average squared velocity. This is called the *root mean square* (RMS) velocity and, although it is not strictly the average speed, its statistical definition is close enough for our purposes. Thus, from Eq. 9.21

$$v_{RMS} = \sqrt{\frac{3k_BT}{m}} \tag{9.23}$$

where T is in K and m is the mass of a single molecule (or atom) in kilograms.

EXAMPLE 9-4

If we consider air to be made up largely of diatomic nitrogen molecules, N_2, what is their RMS velocity at 27° C? One nitrogen atom has a mass of $14 \times 1.67 \times 10^{-27}$ kg, and the mass of N_2 is twice that.

Solution From Eq. 9.23

$$v_{RMS} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23}\ \mathrm{J/K} \times 300\ \mathrm{K}}{28 \times 1.67 \times 10^{-27}\ \mathrm{kg}}} = 515\ \mathrm{m/sec}$$

9.7 MEASUREMENT OF HEAT

The measure of a quantity of heat ΔQ was established by French scientists as the *calorie. One calorie is the quantity of heat required to raise the temperature of 1 g of water by 1° C.* (In the English system the measure is the British thermal unit (BTU), where 1 BTU is the quantity of heat required to raise the temperature of 1 lb of water by 1° F.)

As we discussed in Chapter 5, friction between surfaces causes loss of mechanical energy. However, experience shows us that friction produces heat. Anyone who has used sandpaper on a wooden surface has observed this phenomenon. From the preceding section we recognize that this temperature rise is due to the increased kinetic energy of the molecules. This increase has been produced by the work done on the molecules by the sandpaper. So we see that by our understanding of the nature of temperature we need not restrict the law of energy conservation to frictionless systems; the apparent loss of mechanical energy of the moving system has gone into increased mechanical energy of the molecules.

We may measure how much mechanical energy produces what quantity of heat by a simple experiment (see Fig. 9-5). Suppose we have a paddle wheel driven by a falling weight. The paddle wheel is in a known quantity of water completely insulated from heat flowing in or out. By letting the weight fall with constant velocity and measuring the increased temperature of the water, we may find the mechanical energy equivalent of heat. (*Note*: The weight is allowed to fall at constant speed so that no changes in kinetic energy have to be considered, only changes in potential energy.) The quantity of heat ΔQ is proportional to the temperature rise. This involves the mass of the water m and the *specific heat* of the water c, which is defined as *the amount of heat needed to raise the temperature of 1 g of a substance* (in this case water) 1° C. As mentioned earlier, for water this is 1 cal/g°C. Therefore, the quantity of heat ΔQ required to raise the temperature by ΔT of a mass m of a substance whose specific heat is c, is written as

$$\Delta Q = m\, c\, \Delta T \tag{9.24}$$

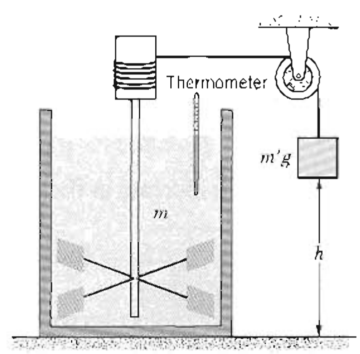


FIGURE 9-5 Diagram of the apparatus for the measurement of the mechanical equivalent of heat.

We can equate this to the loss of potential energy, $\Delta E_p = m'gh$, of the falling weight $m'g$, that is,

$$m'gh = mc\,\Delta T \tag{9.25}$$

For a known amount of ΔE_p and a measured ΔT, the experiment yields the relation

$$4.184\,\mathrm{J} = 1\,\mathrm{cal} \tag{9.26}$$

This relation is called the *mechanical equivalent of heat.*

We may generalize Eq. 9.24 to any substance. On measurement we find, however, that almost all substances have different specific heats. These are not readily calculated but can be determined experimentally and are given by tables in handbooks. The evaluation of the specific heat (or heat capacity) of ideal gases is simpler and is presented next.

9.8 SPECIFIC HEATS OF GASES

If we hold a quantity of gas at a constant volume so that it cannot do work by expanding, then all the heat ΔQ goes into increasing the kinetic energy of the molecules, or

$$\Delta Q = \Delta E_k \tag{9.27}$$

We have discussed c as the specific heat per gram of any substance, but actually we have been talking about solids and liquids. The situation is different for gases because they are compressible. We may define a term C_v as the molar specific heat at constant volume; that is, the specific heat per mole

$$C_v = c_v M \tag{9.28}$$

where M is the mass of a mole of gas and c_v is the specific heat per gram at constant volume. The mass of the gas m is the mass of a mole M multiplied by the number of moles n, that is, $m = Mn$. We may rewrite Eq. 9.24 as $\Delta Q =$ (specific heat per mole) $\times$ (number of moles) $\times (\Delta T)$, or

$$\Delta Q = C_v n\,\Delta T \tag{9.29}$$

The resulting increase in the energy of the molecules may be written as

$$\Delta E_k = (\text{number of molecules}) \times \left(\frac{\Delta E_k}{\text{per molecule}}\right)$$

The number of molecules is equal to the number of moles n, multiplied by Avogadro's number N_A, and, from Eq. 9.21, ΔE_k per molecule $= 3/2k_B\Delta T$. Therefore,

$$\Delta E_k = (nN_A)\left(\frac{3}{2}k_B\,\Delta T\right) \tag{9.30}$$

Substituting Eq. 9.29 for ΔQ and Eq. 9.30 for ΔE_k in Eq. 9.27, we obtain

$$C_v n \, \Delta T = \frac{3}{2} n N_A k_B \, \Delta T$$

or

$$C_v = \frac{3}{2} N_A k_B \qquad (9.31)$$

By definition $k_B = \frac{R}{N_A}$, and Eq. 9.31 becomes

$$C_v = \frac{3}{2} R \qquad (9.32)$$

Note that the conversion factor of Eq. 9.26 will reduce R to a value somewhat easier to remember

$$R = 8314\,\text{J/kmol} - \text{K} = 8.314\,\text{J/mol} - \text{K} \left(\frac{1\,\text{cal}}{4.184\,\text{J}} \right)$$

$$= 1.987 \frac{\text{cal}}{\text{mol K}} \approx 2\,\text{cal/mol} - \text{K}$$

Therefore, Eq. 9.32 for ideal gases, which involve only the translational motion of their atoms (no vibration or rotation as in diatomic gases), yields

$$C_v \approx \frac{3}{2} \times 2\,\text{cal/mole} - \text{K} \approx 3 \frac{\text{cal}}{\text{mol} - \text{K}}$$

This value is expected to hold for all the rare gases that are monatomic, such as helium, neon, and argon. Experiment has proven that the agreement with theory is excellent.

9.9 WORK DONE BY A GAS

Suppose we have a cylinder and a piston with a gas inside, as in Fig. 9-6. Let the cross section of the cylinder be A and the weight of the piston plus a weight resting on it be mg. Suppose the piston was originally at position h_1 and the gas has expanded and pushed it up a distance dx to position h_2. The definition of work, Eq. 5.3, is force times the distance moved in the direction of the force.

$$dW = F\,dx \qquad (5.3)$$

But from the definition of pressure, $P = F/A$, we may substitute for F and obtain

$$dW = PA\,dx$$

Since A dx is the change in volume dV, we may write that work done by a gas as

$$dW = P\,dV \qquad (9.33)$$

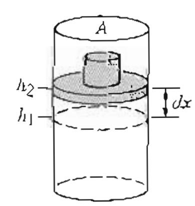


FIGURE 9-6

Note that by the definition of work, if the gas does work by expanding, the work done is positive, whereas if the gas is compressed by the force on the piston, the work done by the gas is negative.

9.10 FIRST LAW OF THERMODYNAMICS

We now have three factors relating energy and the behavior of a gas:

1. Work done on or by a gas $\Delta W = P\Delta V$.
2. The quantity of heat ΔQ that may be added or extracted from the gas.
3. The change in average kinetic energy of the molecules ΔE_k, which we usually call the change in internal energy ΔE.

Because of the interplay of these three terms, it is not meaningful to ask what is the amount of heat in a gas. If heat is added to a gas at constant volume, it all goes into the increase of the internal energy ΔE. If, however, the gas is allowed to expand and do work when heat is added, then the amount of heat available to increase the internal energy depends on the amount of heat that has gone into work. For example, if the gas is allowed to expand and do work while the temperature is held constant ($\Delta T = 0$), then the final internal energy is the same as the initial and $\Delta E = 0$. We may logically write these concepts in the form of the equation

$$\Delta Q = \Delta W + \Delta E \tag{9.34}$$

In Eq. 9.34 ΔQ is taken as positive if heat enters the system (the gas in this case) and as negative if it leaves the system. The work, ΔW, is positive if it is done *by* the system, and negative if done *on* the system. This simple, logical expression bears the ponderous name of the *First Law of Thermodynamics*. It is seen that through the expression for the mechanical equivalent of heat, Eq. 9.26, ΔQ can be expressed as energy, as can ΔW and ΔE. We have then a full statement of the law of conservation of energy in which the E_{out} (energy out) term of the mechanical law, Eq. 5.14, is now included as well as the energy term E_{in}, which may be heat. Furthermore, because work, ΔW, can give rise to changes both in potential and in kinetic energy of a body or system of bodies, all possible mechanical energy terms have been included. Other forms of energy, for example, radiant energy, which will be discussed in a later chapter, are also included in the first law of thermodynamics.

EXAMPLE 9-5

Six thousand calories of heat are added to 2 moles of neon gas at 27° C while it does 4100 J of work. (a) How much does the internal energy of the system increase? (b) What is the final temperature of the gas?

Solution

- (a) First convert calories to joules

$$\Delta Q = 6000\,\text{cal}\left(\frac{4.184\,\text{J}}{1\,\text{cal}}\right) = 2.51 \times 10^4\,\text{J}$$

Use the first law of thermodynamics

$$\begin{aligned}\Delta E &= \Delta Q - \Delta W \\ &= 2.51 \times 10^4\,\text{J} - 0.41 \times 10^4\,\text{J} = 2.10 \times 10^4\,\text{J}\end{aligned}$$

- (b) The relation between the change in the internal energy and the change in temperature is given by Eq. 9.30

$$\Delta E = (nN_A)\left(\frac{3}{2}k_B\,\Delta T\right)$$

Recalling that $N_A\,k_B = R$, we write

$$\Delta E = \frac{3}{2}nR\,\Delta T$$

Therefore

$$\begin{aligned}\Delta T &= \frac{2\,\Delta E}{3\,nR} \\ &= \frac{2 \times 2.10 \times 10^4\,\text{J}}{3 \times 2\,\text{mol} \times 8.314\,\text{J/mol} - \text{K}} \\ &= 842\,\text{K or }^\circ\text{C}\end{aligned}$$

$$T_{\text{final}} = 27^\circ\,\text{C} + 842^\circ\,\text{C} = 869^\circ\,\text{C}$$

SUPPLEMENT 9-1: MAXWELL-BOLTZMANN STATISTICAL DISTRIBUTION

In this chapter we have derived the mean square velocity $\overline{v^2}$ of a large number of atoms or electrons, Eq. 9.21. We did not address the question of how these square velocities are distributed. That is, how many atoms have a square velocity twice the mean or one half the mean. This problem is in the realm of *statistical mechanics*, the science of the application of the first principles of physics to a large number of bodies. The rigorous solution is beyond our interest and is too difficult to present here. However, Professor Richard Feynman has presented a conceptual solution that we will give.

Suppose we have a very tall glass tube with a column of gas going to a great height such as into our upper atmosphere—but, unlike our atmosphere, the temperature is the same at all heights. The problem will be to derive the law of the decrease in density of the atmosphere as the height increases. Let n be the number of moles of gas in volume V at pressure P. From the ideal gas law Eq. 9.6 we write $P = nRT/V$. We recall that $R = k_B N_A$, therefore, $P = (nN_A/V)(k_B T)$. The ratio nN_A/V represents the number of molecules per unit volume N. Thus the pressure is proportional to the number of molecules per unit volume because the temperature is constant; that is, $P = Nk_B T$.

The pressure is higher the lower we measure it, for at any point the gas must support all the gas above it. Consider a small cylindrical volume of gas of cross-sectional area $A = 1\ \text{m}^2$ and width dy at a height y (see Fig 9-7). The vertical force from below on this gas is P_y, because $F = P_y A$ and we have taken $A = 1\ \text{m}^2$. The vertical force from above at a height $y + dy$ is less than P_y by the weight of the molecules in the section between y and $y + dy$. The total number of molecules in this region is the number per unit volume N times the volume dy (since A was taken as unity). Each molecule has a weight of mg, so the difference in pressure is

$$P_{y+dy} - P_y = dP = -mg\,N\,dy. \qquad (9.35)$$

Ludwig Boltzmann (1844–1906).

We have seen that $P = N\,k_B T$. Because T is constant, the pressure depends only on N. Differentiating P with respect to N we obtain

$$dP = k_B T\,dN \qquad (9.36)$$

Equating Eq. 9.36 to Eq. 9.35, we have an equation that can be integrated to find N

$$k_B T\,dN = -mg\,N\,dy$$

or

$$\frac{dN}{N} = -\frac{mg}{k_B T}\,dy$$

Integrating

$$\int_{N_0}^{N} \frac{dN}{N} = -\frac{mg}{k_B T}\int_0^y dy$$

which yields

$$N = N_0\,e^{-mgy/kT} \qquad (9.37)$$

where the constant N_0 is the value of N at $y = 0$ and this value of y may be at any predetermined level.

We see that the numerator of the argument of the exponential in Eq. 9.37 contains the potential energy per molecule, and we may write that the density at any point is

$$N = N_0\,e^{-E_p/k_B T} \qquad (9.38)$$

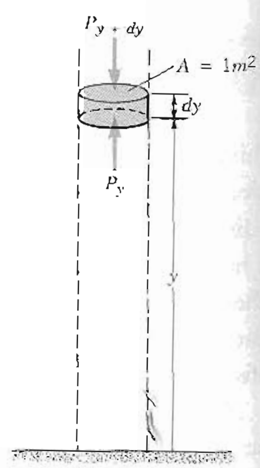


FIGURE 9-7

The source of the potential energy need not be gravitational. For example, the potential energies of electrons at different distances from their nuclei have an electrical origin, as we will see in Chapter 14.

In Chapter 5 we saw that as a particle moves in a gravitational field (or any other conservative field) the potential energy of the particle may change. However, the total energy, which is the sum of the potential and kinetic energies, remains constant. Thus it is reasonable to assume that if we start with a given number of molecules having a certain value for the potential energy, the number having that particular value for the total energy, sometime later, will be the same. This means that Eq. 9.38 can be generalized to represent the number of molecules having a certain value for the total energy E; that is,

$$N = N_0 e^{-E/k_B T} \tag{9.39}$$

Eq. 9.39 is known as the *Maxwell-Boltzmann statistical distribution of energy*.

In general, if $E_2 - E_1$ is some energy difference between energy state 1 and a higher energy state 2 of a particle, the ratio of the number occupying the higher energy state to the lower energy state is given by

$$\frac{N_2}{N_1} = e^{-(E_2-E_1)/k_B T} \tag{9.40}$$

We can use Eq. 9.39 to answer another important question, namely, what fraction of the total number of particles has energy equal to or greater than a certain value E_i? Because Eq. 9.39 represents the number of particles with energy E, the number of particles with energy between E and $E + dE$ will be proportional to $N_0 e^{-E/k_B T}\, dE$. It follows that the fraction of particles with energies greater than or equal to E_i is

$$\begin{aligned}
\text{Fraction } (E \geq E_i) &= \frac{\int_{E_i}^{\infty} N_0 e^{-E/k_B T}\, dE}{\int_0^{\infty} N_0 e^{-E/k_B T}\, dE} \\
&= \frac{\int_{E_i}^{\infty} e^{-E/k_B T} \left(\frac{dE}{k_B T}\right)}{\int_0^{\infty} e^{-E/k_B T} \left(\frac{dE}{k_B T}\right)} \\
&= \frac{e^{-E/k_B T}\Big|_{E_i}^{\infty}}{e^{-E/k_B T}\Big|_0^{\infty}} \\
&= e^{-E_i/k_B T}
\end{aligned} \tag{9.41}$$

The term of Eq. 9.41 is often referred to as the *Boltzmann factor*.

PROBLEMS

9.1 The molecular weights of sodium and chlorine are 22.99 g/mole and 35.45 g/mole, respectively. How many molecules of sodium chloride (ordinary salt) are there in 100 g of salt?

9.2 The density of copper is $9\,g/cm^3$, its molecular weight is 64 g/mole. What is the number of copper atoms in $1\,m^3$?

Answer: 8.47×10^{28}.

9.3 (a) What is the body temperature of a healthy person on a Celsius clinical thermometer? (b) If the person had a temperature of 102° F what would the thermometer read?

9.4 At what temperature will Fahrenheit and Celsius thermometers read the same value?

Answer: −40°.

9.5 A gas bubble rises from the bottom of a lake to the surface. If the pressure at the bottom is three times atmospheric pressure and the temperature is 4° C while near the surface the temperature is 24° C, what is the ratio of the volume of the bubble just before it reaches the surface to its volume at the bottom of the lake?

9.6 In an ultrahigh vacuum system, the pressure can be lowered to 10^{-10} torr (1 torr = 1/760 atm). How many molecules are there in a vacuum chamber of volume $8 \times 10^6\,cm^3$ if the pressure is 10^{-10} torr and the temperature is 27° C? 1 atm = $1.01 \times 10^5\,N/m^2$.

Answer: 2.57×10^{13}.

9.7 A gas tank contains 10 kg of oxygen at a pressure of $10^7\,N/m^2$ and a temperature of 27° C. As a result of a leak, the pressure drops to $5 \times 10^6\,N/m^2$ and the temperature decreases to 7° C. (a) What is the volume of the tank? (b) How much oxygen has leaked out? The molecular weight of oxygen is 32 g/mole.

9.8 The interior of the sun is at a temperature of about 1.5×10^8 K. The energy is created by the fusion of hydrogen atoms when they collide. In developing the technology for a fusion reactor we simulate this fusion reaction by accelerating protons (hydrogen nuclei) and letting them strike fixed-target hydrogen atoms. What must be the velocity of the protons to simulate 1.5×10^8 K?

9.9 One gram of Ne gas (atomic weight 20.2 g/mole) is in a sealed flask at room temperature, 27° C. If 10 calories of heat are added to the gas, what is the v_{RMS} of the molecules?

Answer: 6.72×10^2 m/sec.

9.10 The temperature of a gas in a closed container at 27° C is raised to 327° C. By what multiple has v_{RMS} changed?

9.11 The temperature of a room 7 m × 5 m × 3 m is 27° C. (a) How much energy is contained in the air of that room? (b) If that energy could be converted to electrical energy, for how long could a 100-W bulb be lit? Assume the air behaves as an ideal gas.

9.12 A 300-W immersion heater is used to heat a cup of water. If the cup contains 150 g of water at 27° C and 80% of the heater energy is absorbed by the water, how long will it take for the water to begin to boil?

Answer: 191 sec.

9.13 A 1000-W heater is used to heat the room of problem 9.11. If the molar specific heat of air is 5 cal/mole-K, how long will it take to raise the temperature of the room from 60° F to 70° F? Assume that the quantity, volume, and pressure of air do not change.

9.14 How many calories of heat must be added to 0.5 g of neon gas at constant volume to raise its temperature from 27° C to 127° C? The molecular weight of neon is 20.2 g/mole.

Answer: 7.40 cal.

9.15 To heat a certain quantity of gas from 27° C to 127° C requires 500 cal when its volume is kept constant. By how much does its internal energy change? How much work could the gas do in cooling back to 27° C?

9.16 A 20-g bullet is shot into a ballistic pendulum with a velocity of 1000 m/sec (see Fig. 9-8). The mass of the wooden block is 2 kg. If the bullet remains embedded in the block and 80% of the energy lost in the collision is absorbed as heat by the bullet, what is the increase in the temperature of the bullet? The specific heat of the bullet is 0.1 cal/g-°C.

Answer: 946° C.

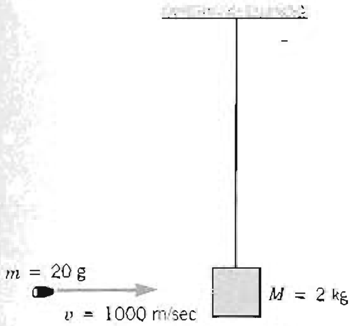


FIGURE 9-8 Problem 9.16.

9.17 Ten moles of an ideal gas at a pressure of 8 atm and with volume 10^{-2} m^3 are allowed to expand isothermally (constant temperature) until the volume doubles. What is the work done by the gas? (1 atm = 1.01×10^5 N/m^2).

Answer: 5.6×10^3 J.

9.18 Use the ideal gas law, Eq. 9.6, and the first law of thermodynamics, Eq. 9.34, to show that the molar specific heat for a process at constant pressure $C_p = C_v + R$, where C_v is the molar specific heat at constant volume and R is the universal gas constant.

9.19 The initial pressure and volume of 0.1 moles of argon gas are 1 atm (1.01×10^5 N/m^2) and 1 liter (10^{-3} m^3) (see Fig. 9-9). The gas is heated at constant volume until the pressure rises to 4 atm (path A). The gas is then allowed to expand along path B until the pressure drops to 1 atm. The gas is finally cooled down at constant pressure until it returns to its initial state (path C). (a) Find the temperature of the gas at the end of each process (points 1, 2, and 3). (b) Find the internal energy of gas at points 1, 2, and 3. (c) Calculate the work done in each process. (d) Calculate the heat entering or leaving the gas during each process.

Answer: (a) 122 K, 487 K, 609 K, (b) 152 J, 607 J, 760 J, (c) 0 J, 1.01×10^3 J, -4.04×10^2 J, (d) 455 J, 1163 J, −1012 J (leaving the gas).

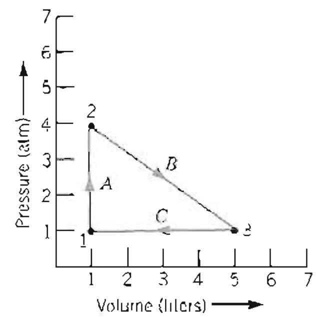


FIGURE 9-9 Problem 9.19.

9.20 A mole of an ideal gas is taken from state A to state C along the path ABC (see Fig. 9-10). (a) If 1000 cal of heat flow into the gas and the gas does 2100 J of work, what is the change in the internal energy of the gas? (b) When the gas is returned from C to A along the path CDA, 700 cal of heat flow out of the gas. How much work is done on the gas? (c) What is the change in the temperature of the gas when it is brought back from C to A? (d) If the pressure of the gas in state A is 2 atm, what is the difference in the volume of the gas between states D and A?

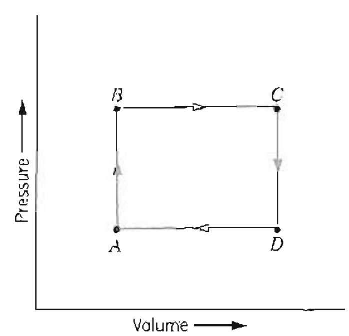


FIGURE 9-10 Problem 9.20.

10.1 INTRODUCTION

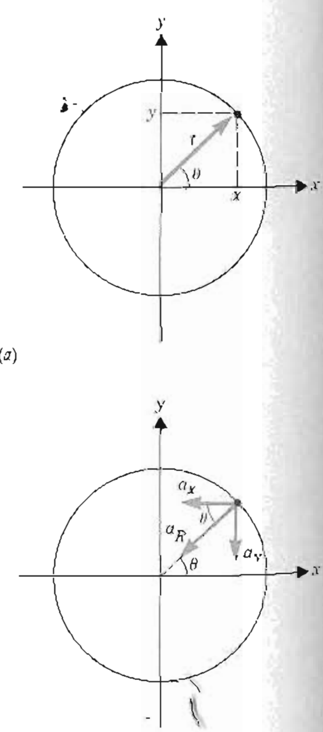
In this chapter we will develop the concepts of oscillatory motion. When a block attached to a spring is set into motion, its position is a periodic function of time. Similarly, in Chapter 7 when we considered the motion of a particle rotating in a circle, we saw that the position coordinates were oscillatory functions of time. Specifically, we showed (Fig. 10-1*a*) that the components of a position vector **r** making an angle θ with the x axis were

$$x = r\cos\theta$$
$$y = r\sin\theta \tag{10.1}$$

and that the components of the acceleration, second derivatives with respect to time of these coordinates, were (Fig. 10-1*b*)

$$a_x = \frac{d^2x}{dt^2} = -r\omega^2\cos\theta$$
$$a_y = \frac{d^2y}{dt^2} = -r\omega^2\sin\theta \tag{10.2}$$

As the angle θ goes from 0° to 360° the components of the position vector and of the acceleration vary in value as the sine or cosine functions and go through a reversal of sign; in other words, their values oscillate sinusoidally. In this chapter we will derive some properties of this oscillatory motion with essentially the identical mathematical method that generated Eqs. 10.1 and 10.2.



(*a*)

(*b*)

FIGURE 10-1

10.2 CHARACTERIZATION OF SPRINGS

Robert Hooke (1635–1703), an English scientist, was the first to elucidate the behavior of an elastic body such as a spring. He found that the extension or compression x of an elastic body is proportional to the applied force F or

$$F \propto x$$

This simple relationship is now known as Hooke's law. We introduce a proportionality constant to create an equality. This constant has the symbol k and is called the *force constant* or *spring constant*

$$F = kx \tag{10.3}$$

In the case of a spring, the value of the constant k characterizes the strength (or stiffness) of the spring—a spring with a large k is stronger, or stiffer, than one with a small k. We may readily measure the value of k by simply hanging a weight on the spring and measuring how much it stretches. For the measurement to be valid, the spring must return to its original length when the weight is removed. The extension of the spring in this case is within its ***elastic limit***. If a spring is stretched beyond its elastic limit it will deform permanently and Hooke's law is not obeyed.

10.3 FREQUENCY AND PERIOD

Suppose we have a periodic event, that is, one that occurs regularly with time such as the rising of the sun. We know that it occurs once each day; that is, its *frequency* ν is one event per day, and it has dimensions of $(\text{time})^{-1}$ because event is dimensionless. We use another quantity, the time between periodic events, known as the *period* with symbol T. The period of the sun's rising is 1 day per event, and obviously has dimensions of time. It is seen that period and frequency are reciprocals of each other.

$$\nu = \frac{1}{T} \tag{10.4}$$

If, for example, the time between periodic events is $T = 0.2$ sec, then the number of events per second, the frequency ν, will be

$$\nu = \frac{1}{0.2\ \text{sec/event}} = 5\ \text{events/sec}$$

As we have indicated, frequency has units of $(\text{time})^{-1}$ or events per second. One event per second is called one hertz, abbreviated Hz.

If a point particle moves on a circle as in Fig. 10-1*a*, its position vector from the center of the circle to the particle has the magnitude of the radius. This is often called the *radius vector* of the point, or simply the radius vector. As the particle moves, the radius vector rotates and if the particle moves with constant speed in the counterclockwise direction, we say that it rotates with a constant positive rotational speed ω. Then θ is a function of time, and from Eq. 7.8

$$\theta = \omega t \tag{10.5}$$

We can use this result to express the coordinates x and y of the rotating particle in Fig. 10-1 as explicit functions of time and the frequency of rotation. Substituting Eq. 10.5 for θ in Eq. 10.1, we obtain

$$\begin{aligned} x &= r\cos\omega t \\ y &= r\sin\omega t \end{aligned} \tag{10.6}$$

The angular speed ω is related to the frequency of rotation ν rather simply. In every rotation θ changes by 2π rad. If the particle performs ν rotations in 1 sec, then θ will change by $2\pi\nu$ rad every second. By definition, ω is the change in θ per unit time (per second). We conclude that

$$\omega = 2\pi\nu \qquad (10.7)$$

Substituting Eq. 10.7 for ω in Eq. 10.6 we may write

$$x = r\cos 2\pi\nu t$$
$$\qquad (10.8)$$
$$y = r\sin 2\pi\nu t$$

10.4 AMPLITUDE AND PHASE ANGLE

Fig. 10-2 shows a plot of $\sin\theta$ versus θ. We see that the value of $\sin\theta$ oscillates between $+1$ and -1. The maximum value of the magnitude of this oscillation is called the amplitude. In Fig. 10-2, the amplitude is 1. If $\sin\theta$ were multiplied by a constant A, then A would be the amplitude in the expression $A\sin\theta$. Suppose instead of $\sin\theta$ we plot the function $\sin(\theta + \pi/4)$. We see that when $\theta = 0$ the function has the value of $\sin\pi/4$ and thereafter attains all values of $\sin\theta$ at an angle $\pi/4$ earlier, as shown in Fig. 10-3*a*. If, on the other hand, we plot the function $\sin(\theta - \pi/4)$ we see in Fig. 10-3*b* that it starts later than the $\sin\theta$ function. The general form for a function to describe a body undergoing sinusoidal oscillations, such as the one illustrated in Fig. 10-1 is

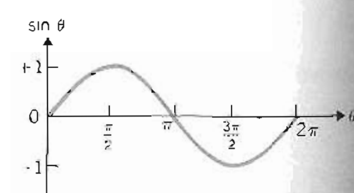


FIGURE 10-2

$$A\sin(\theta + \phi)$$

or, because $\theta = \omega t$, this may be written as

$$A\sin(\omega t + \phi) \qquad (10.9)$$

where ϕ is called the *phase angle* and its sign may be positive or negative. Note that if $\phi = \pi/2$, then $\sin(\theta + \pi/2) = \cos\theta$, which may be seen by sketching a $+\pi/2$ phase shift on Fig. 10-3*a*. The motion described by Eq. 10.9 is often referred to as *simple harmonic motion*.

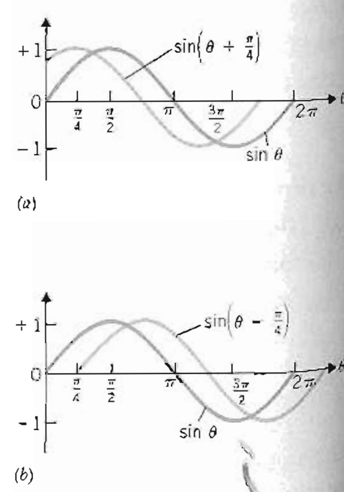


FIGURE 10-3

10.5 OSCILLATION OF A SPRING

Suppose a body of mass m is connected to a massless spring, with a spring constant k, and the body is free to oscillate on a frictionless surface as in Fig. 10-4. At its rest, or equilibrium position, the position coordinate is $x = 0$, indicated in Fig. 10-4*a*. If the body is pushed to compress the spring a distance x_0, Fig. 10-4*b*, or pulled to stretch it a distance x_0, Fig. 10-4*c*, and then released, the body will then begin to oscillate.

We may calculate its subsequent motion from Newton's second law

$$\mathrm{F} = m\mathrm{a}$$

Eq. 10.3 gives an expression of Hooke's law, $F = kx$, for a spring. However, this is the external force needed to compress or to stretch the spring. By Newton's third law of action and reaction, if you pull on a spring with force F it pulls in the opposite direction with force $-F$. Thus, the force that the spring exerts on the body is $-kx$. Because the acceleration is not constant (the force on the body depends on displacement x from equilibrium), we use the fundamental definition $a_x = d^2x/dt^2$. Applying Newton's second law to the body, we obtain

$$-kx = m\frac{d^2x}{dt^2} \tag{10.10}$$

This is a second-order differential equation; although there are straightforward mathematical techniques for its solution, we will simply guess a solution and substitute it into Eq. 10.10 to see if an equality is maintained. Such a procedure can verify that the guessed function is a solution but does not prove that it is the only solution. We may try as our guess the function introduced earlier, Eq. 10.9, to describe a body undergoing sinusoidal oscillations. Our guess at a solution will be

$$x = A\sin(\omega t + \phi) \tag{10.9}$$

We may substitute this directly into the left side of Eq. 10.10, but for the right side we need its second derivative

$$\frac{dx}{dt} = A\frac{d}{dt}\sin(\omega t + \phi) = A\omega\cos(\omega t + \phi)$$

$$\frac{d^2x}{dt} = A\omega\frac{d}{dt}\cos(\omega t + \phi) = -A\omega^2\sin(\omega t + \phi) \tag{10.11}$$

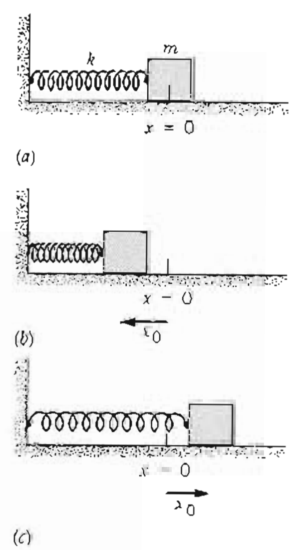


FIGURE 10-4

Substituting Eqs. 10.9 and 10.11 into Eq. 10.10 obtains

$$-kA\sin(\omega t + \phi) = -m\omega^2 A\sin(\omega t + \phi)$$

Cancelling obtains

$$k = m\omega^2$$

and

$$\omega = \sqrt{\frac{k}{m}} \tag{10.12}$$

Therefore, Eq. 10.9 is a solution when the constants have the relation of Eq. 10.12. Using Eq. 10.7, that $\omega = 2\pi\nu$, we immediately obtain the frequency of oscillation

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{10.13}$$

and the period

$$T = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{k}}$$

To complete the solution of the problem, we must determine the value of the amplitude A and of the phase angle ϕ in the expression for x, Eq. 10.9. This is done by specifying the ***boundary conditions***, that is, the behavior of the body at some time such as, $t = 0$. For example, the body in Fig. 10-4 can be set into oscillation by initially stretching the spring a certain distance $x = x_0$ as shown in Fig. 10-4c and then releasing it. That is, at $t = 0$.

$$x = x_0 \tag{10.14}$$

$$v_x = 0 \tag{10.15}$$

The first condition, Eq. 10.14, is satisfied by setting $x = x_0$ and $t = 0$ in Eq. 10.9. This yields

$$x_0 = A\sin\phi \tag{10.16}$$

To impose the second condition, we must first determine the velocity of the body as a function of time. This is done as follows:

$$v_x = \frac{dx}{dt} = \frac{d}{dt}A\sin(\omega t + \phi)$$

$$v_x = \; = A\omega\cos(\omega t + \phi) \tag{10.17}$$

The second condition, Eq. 10.15, will be satisfied by setting $v_x = 0$ and $t = 0$ in Eq. 10.17; that is,

$$0 = A\omega\cos\phi \tag{10.18}$$

The amplitude and the phrase angle can now be found by solving simultaneously Eqs. 10.16 and 10.18. If we divide Eq. 10.18 by Eq. 10.16, we obtain

$$\frac{0}{x_0} = \frac{A\omega\cos\phi}{A\sin\phi}$$

or

$$\cot\phi = 0$$

hence

$$\phi = \frac{\pi}{2} \tag{10.19}$$

Substituting Eq. 10.19 for ϕ in Eq. 10.16 yields the result

$$x_0 = A\sin\frac{\pi}{2} = A \tag{10.20}$$

Thus, we see that the amplitude, in this case, is equal to the initial displacement of the body from its equilibrium position, and the phase angle is $\pi/2$ rad. We should note that other boundary conditions will yield different values for A and ϕ.

We can use the facts that $\sin(\theta + \pi/2) = \cos\theta$ and $\cos(\theta + \pi/2) = -\sin\theta$, to eliminate ϕ from the expressions Eq. 10.9 for x and Eq. 10.17 for υ_x, which now become

$$x = A\cos\omega t \tag{10.21}$$

$$\upsilon_x = -A\omega\sin\omega t \tag{10.22}$$

We see by Eq. 10.22 that immediately after release from its stretched position to the right, the velocity of the body is toward the left, hence the negative sign. When the argument of the sine, ωt, exceeds π, then the sine function becomes negative and the velocity is positive, or toward the right. We also see from Eq. 10.22 that because the maximum value the sine function may have is ± 1, then the maximum velocity of the block is

$$\upsilon_{max} = \pm A\omega = \pm A\sqrt{\frac{k}{m}} \tag{10.23}$$

Furthermore, because $\sin\omega t = 1$ when $\omega t = \pi/2$ and -1 when $\omega t = 3\pi/2$, insertion of these values into Eq. 10.21 shows that the maximum velocity occurs when $x = 0$ or at the midpoint of oscillation. It is instructive to compare a plot of Eq. 10.22 with one of Eq. 10.21 as shown in Fig. 10-5. Note that the amplitude of the displacement A and the maximum value of the velocity $A\omega$ are not the same because ω may be equal to or greater or smaller than unity. Figure 10-5 is a plot with $A\omega \approx 1.2A$. It can be seen that the velocity is maximum when the displacement is zero and zero when the displacement is maximum. The physical significance of this result will become evident when we discuss the energy associated with oscillations in the next section.

We may examine the behavior of the acceleration by taking the time derivative of the velocity in Eq. 10.22.

$$a_x = \frac{d\upsilon_x}{dt} = -A\omega\frac{d}{dt}\sin\omega t$$

$$a_x = -A\omega^2\cos\omega t \tag{10.24}$$

We plot a_x versus θ and compare it with the displacement (x value). Because ω for Fig. 10-5 was taken as 1.2, ω^2 is 1.4, so the maximum value of a_x will be 1.4 times the amplitude of x. The acceleration and displacement curves are plotted in Fig. 10-6. Here we see that the acceleration, although also a cosine curve but with a different amplitude, is a reflection about the θ axis of the displacement. That is, when the displacement is maximum in the positive direction, the acceleration is maximum in the negative direction. Furthermore, when the displacement is zero, so is the acceleration. This relation can be seen physically in Fig. 10-7. When the body is displaced to the

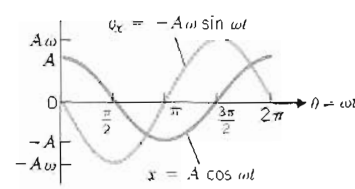


FIGURE 10-5 Plot of x and v as functions of time for ω greater than unity.

right (positive x direction) a distance A and then released (Fig. 10-7a), the acceleration is in the negative x direction. The acceleration is also a maximum at this position. This is evident because $F = ma$ and $F = -kx$. Therefore, $-kx = ma$ and a is maximum when x is maximum and x and a have opposite signs. When the body reaches $x = 0$ (Fig. 10-7b), the acceleration is 0, but as it passes to the left the displacement x becomes negative and the acceleration becomes positive or toward the right (Fig. 10-7c). This behavior is shown schematically in Fig. 10-6. At $\theta = \pi/2$, which corresponds to zero displacement, the acceleration goes to zero. As the displacement becomes negative ($\pi/2 < \theta < 3\pi/2$) the acceleration becomes positive. We also note from Fig. 10-6 that the two maxima of acceleration occur at $\theta = 0$ or $\pi (\omega t = 0$ or $\pi)$. The maximum values of the acceleration at these two positions are, from Eq. 10.24,

$$a_x = -A\omega^2 \cos 0 = -A\omega^2$$

$$a_x = -A\omega^2 \cos \pi = A\omega^2$$

or

$$a_{\text{max}} = \pm A\omega^2 \tag{10.25}$$

EXAMPLE 10-1

Show that $x = A\cos(\omega t + \phi)$ is also a solution of Eq. 10.10.

Solution

$$-kx = m\frac{d^2x}{dt^2} \tag{10.10}$$

Take the second derivative and substitute into the right side of Eq. 10.10.

$$\frac{dx}{dt} = A\frac{d}{dx}\cos(\omega t + \phi) = -A\omega \sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega\frac{d}{dx}\sin(\omega t + \phi) = -A\omega^2 \cos(\omega t + \phi)$$

Substituting in Eq. 10.10,

$$-kA\cos(\omega t + \phi) = -mA\omega^2 \cos(\omega t + \phi)$$

After cancelling the cosine term from both sides of this equation, we obtain

$$\omega = \sqrt{\frac{k}{m}}$$

Thus, $A\cos(wt + \phi)$ is a solution of Eq. 10.10 for this value of ω.

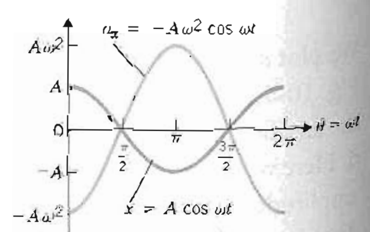


FIGURE 10-6 Plot of x and a as functions of time for ω greater than unity.

EXAMPLE 10-2

A given spring stretches 0.1 m when a force of 20 N pulls on it. A 2-kg block attached to it on a frictionless surface as in Fig. 10-4 is pulled to the right 0.2 m and released. (a) What is the frequency of oscillation of the block? (b) What is its velocity at the midpoint? (c) What is its acceleration at either end? (d) What are the velocity and acceleration when $x = 0.12$ m, on the block's first passing this point?

Solution First we must determine the spring constant k.

$$k = \frac{F}{x} = \frac{20\,\text{N}}{0.1\,\text{m}} = 200\,\text{N/m}$$

- (a) We may then calculate ω from Eq. 10.12

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\,\text{N/m}}{2\,\text{kg}}} = 10\,\text{rad/sec}$$

Because

$$\omega = 2\pi\nu$$

$$\nu = \frac{\omega}{2\pi} = \frac{10\,\text{rad/sec}}{2\pi} = 1.6\,\text{Hz}$$

- (b) The velocity is a maximum when $x = 0$, that is, at the midpoint. Therefore, from Eq. 10.23, (recall, as shown earlier in this section, that when a block is initially displaced a distance x_0 from its equilibrium position and then released, the amplitude of the motion $A = x_0$)

$$v = v_{\text{max}} = \pm A\omega = \pm(0.2\,\text{m})(10\,\text{rad/sec}) = \pm 2\,\text{m/sec}$$

- (c) The acceleration is a maximum at the two extremes of the motion. Therefore, from Eq. 10.25

$$a_{\text{max}} = \pm A\omega^2 = \pm(0.2\,\text{m})(10\,\text{rad/sec})^2 = \pm 20\,\text{m/sec}^2$$

- (d) To determine the block's velocity and acceleration at some arbitrary value of x, we need to know the angle ωt at that position. In this problem, $x = 0.12$ m. We use the relation

$$x = A\cos\omega t \qquad (10.21)$$

$$\omega t = \text{arc}\cos\frac{x}{A} = \text{arc}\cos\frac{0.12\,\text{m}}{0.2\,\text{m}} = 53°$$

Then we may substitute into Eqs. 10.22 and 10.24

$$v = -A\omega\sin\omega t \qquad (10.22)$$

$$= -(2.0\,\text{m})(10\,\text{rad/sec})\sin 53° = -1.6\,\text{m/sec}$$

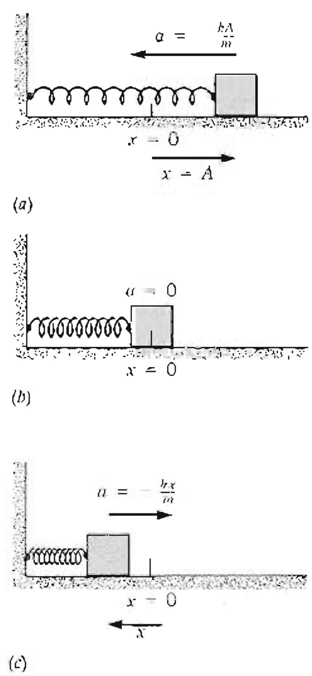


FIGURE 10-7

(Moving toward the left)

$$a = -A\omega \cos \omega t \quad (10.24)$$

$$= -(2.0\,\text{m})(10\,\text{rad/sec}^2)^2 \cos 53^\circ = -12\,\text{m/sec}^2$$

(Accelerating toward the left)

10.6 ENERGY OF OSCILLATION

In Chapter 5 we saw that when an object is raised to a height y in the gravitational field on its descent the gravitational force is capable of doing work on the object. Because of this, there is associated with the object at a height y a potential energy $E_p = mgy$. We defined the potential energy as the work done in raising the object to that height. An analogous situation occurs here. When a body attached to a spring is displaced from its equilibrium position ($x = 0$), the spring is potentially capable, on the release of the body, to do work on the body. We can therefore associate with the spring-body system a potential energy E_p. This potential energy will be the work done in stretching or compressing the spring.

When the force **F** and the displacement $d\mathbf{x}$ are in the same direction, work was defined as the product of the magnitudes of the force and the displacement (see Eq. 5.1), that is,

$$dW = F\,dx$$

or

$$W = \int_0^x F\,dx$$

From Hooke's law, Eq. 10.3, the force needed to compress or stretch a spring is $F = kx$; thus

$$W = k\int_0^x x\,dx$$

$$W = \frac{1}{2}kx^2$$

The potential energy of the spring-body system, when the body is displaced a distance x from its equilibrium position, is therefore

$$E_p(\text{spring}) = \frac{1}{2}kx^2 \quad (10.25)$$

This equation was derived on the assumption that the spring was initially in its equilibrium position, $x = 0$. This assumption is not necessary. If the spring is initially in a position x_1 and is compressed or stretched to position x_2, the work done is as before

$$W = k \int_{x_1}^{x_2} x\, dx$$

$$= \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 \qquad (10.26)$$

Note that because the displacement is squared the potential energy of a spring is the same whether it is stretched or compressed an equal distance x from its relaxed position.

If there is no friction we may expect, as was the case with the gravitational force, that the total mechanical energy, kinetic plus potential, will remain constant as the body oscillates. This can be shown rather simply. By the work-energy theorem, Eq. 5.9, the work done by the spring, as the body moves between two arbitrary displacements x_1 and x_2, is equal to the change in the kinetic energy of the body; that is,

$$\int_{x_1}^{x_2} F_{\text{spring}}\, dx = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \qquad (10.27)$$

where v_1 and v_2 are the velocities of the body at x_1 and x_2, respectively. The force exerted by the spring on the body is $F_{\text{spring}} = -kx$. Substituting this for F_{spring} in Eq. 10.28, and integrating the left side of the equation we obtain

$$\int_{x_1}^{x_2} kx\, dx = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right)$$

Eq. 10.28 becomes

$$-\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Rearranging terms, we obtain

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2 \qquad (10.28)$$

Because x_1 and x_2 are arbitrary points we conclude that the total energy

$$E_{\text{total}} = E_p(\frac{1}{2}kx^2) + (\frac{1}{2}mv^2)$$

remains constant as the body oscillates.

Note that when the spring is stretched, $x = A$. Before the object is released it has no velocity and $1/2\, kA^2$ is the total energy of the system; after it is released the energy remains constant because energy neither enters nor leaves the system.

In Section 10.5 we saw that the velocity was a maximum when the displacement was zero, and it was zero when the displacement was a maximum. This result is intimately tied to the fact that the total mechanical energy of the system remains constant. Therefore, the kinetic energy (and hence the velocity) will be a maximum when the potential energy is a minimum, that is, when x is zero. The kinetic energy will be a minimum (zero) when the potential energy is a maximum, that is, when $x = A$.

EXAMPLE 10-3

The block of Example 10-2 is released from a position of $x_1 = A = 0.2$ m as before. (a) What is its velocity at $x_2 = 0.1$ m? (b) What is its acceleration at this position?

Solution

- (a) The velocity at x_2 can be found with the conservation of energy equation, Eq. 10.29

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2$$

Solving for v_2, noting that $v_1 = 0$, we obtain

$$v_2 = \left[\frac{k(x_1^2 - x_2^2)}{m}\right]^{1/2}$$

$$= \left[\frac{200\,\text{N/m}[(0.2\,\text{m})^2 - (0.1\,\text{m})^2]}{2\,\text{kg}}\right]^{1/2}$$

$$= 1.73\,\text{m/sec}$$

- (b) We may find the acceleration at this position by using Newton's second law

$$F = ma$$

$$-kx = ma$$

$$a = -\frac{kx}{m} = -\frac{(200\,\text{N/m})(0.1\,\text{m})}{2\,\text{kg}} = -10\,\text{m/sec}^2$$

PROBLEMS

10.1 Show that $x = A \sin \omega t + B \cos \omega t$, where A and B are arbitrary constants, is a solution of Eq. 10.10.

10.2 The position of a particle undergoing oscillations is given by $x = 25 \sin(3\pi t + \pi/5)$, where x is in centimeters and t in seconds. Find (a) the frequency of the motion, (b) the amplitude of the motion, (c) the maximum velocity of the particle, (d) the maximum value of the acceleration of the particle, (e) the position, the velocity, and the acceleration of the particle at $t = 0$.

10.3 The same block on the same spring as in Example 10-2 is released after being pulled 0.2 m to the right. Find its position, velocity, and acceleration 0.1 sec after being released.

10.4 A small block attached to a spring is oscillating horizontally on a frictionless surface with an amplitude of 0.12 m. When it is at the position $x = 0.05$ m its velocity is 2 m/sec. (a) What is its frequency of oscillation? (b) What is its position when its velocity is 1 m/sec?

Answer: (a) 2.92 Hz, (b) 0.107 m.

10.5 An oscillating block of mass 250 g takes 0.15 sec to move between the endpoints of the motion, which are 40 cm apart. (a) What is the frequency of the motion? (b) What is the amplitude of the motion? (c) What is the force constant of the spring?

10.6 When a mass of 0.2 kg is suspended from a spring, it stretches 0.04 m. The mass is pulled down an additional distance 0.1 m from its equilibrium position and released. (a) What is the spring constant? (b) What is the period of oscillation? (c) What is the frequency of oscillation? (d) What will be the maximum velocity?

Answer: (a) 49 N/m, (b) 0.40 sec, (c) 2.5 Hz, (d) 1.57 m/sec.

10.7 (a) Write down the equation for the position y (measured from the equilibrium position) of the mass in problem 10-6. (b) What is the equation for the velocity of the mass as a function of time? (c) What is the equation for the acceleration of the mass as a function of time? Take positions below the equilibrium point as positive.

10.8 (a) How long after being released is the position of the mass in problems 10-6 and 10-7 equal to 0.05 m? (b) What is the velocity of the mass when $y = 0.05$ m? (c) What is the acceleration of the mass when $y = 0.05$ m?

Answer: (a) 6.7×10^{-2} sec, (b) -1.36 m/sec, (c) -12.25 m/sec^2.

10.9 The position of an oscillating particle is given by $x = A\sin(\omega t + \phi)$, Eq. 10.9. A particle of mass $m = 0.5$ kg is connected to a spring of force constant $k = 200$ N/m. The particle is initially at rest on a frictionless table. The particle is given an initial velocity of 1.5 m/sec to start oscillating. What is the amplitude of the motion A and the phase angle ϕ?

Answer: 7.5×10^{-2} m, 0 rad.

10.10 Two springs with force constants $k_1 = 100$ N/m and $k_2 = 200$ N/m are connected to opposite ends of a block of mass 3 kg (see Fig. 10-8). (a) If the block is displaced 0.1 m to the right, what is the net force exerted by the springs on the block? The block is released from that position. (b) What are the frequency and the period of the motion? (c) What is the amplitude of the motion? (d) Find an expression for the position of the particle as a function of time.

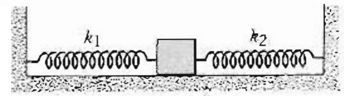


FIGURE 10-8 Problem 10.10.

10.11 A block of mass 2 kg sits on a platform that is oscillating in a vertical plane with an amplitude of 10 cm (see Fig. 10-9). If the frequency of oscillation is 1 Hz, what is the normal force exerted by the platform on the block (a) as they pass the equilibrium point, (b) at the lowest point of the motion, (c) at the highest point of the motion? (d) If the frequency remains constant, at what value of the amplitude will the block and the platform separate?

Answer: (a) 19.6 N, (b) 27.5 N, (c) 11.7 N, (d) 24.8×10^{-2} m.

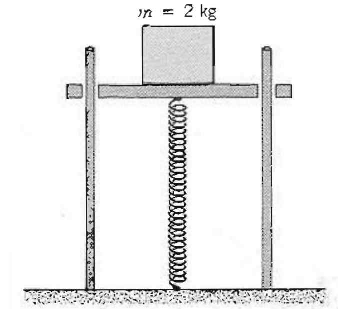


FIGURE 10-9 Problem 10.11.

10.12 A block is oscillating horizontally on a frictionless table with an amplitude of 5 cm. A coin of mass $m = 3$ g is placed on top of the block. The maximum force of friction between the coin and the block is 0.015 N. What is the maximum value of the frequency of oscillation for which the coin will stay on top of the block?

Answer: 1.59 Hz.

10.13 A block is oscillating with an amplitude of 20 cm. The spring constant is 150 N/m. (a) What is the energy of the system? (b) When the displacement is 5 cm, what is the kinetic energy of the block and the potential energy of the spring?

10.14 A 0.25-kg mass is oscillating with a frequency $\nu = 5$ Hz. What is the amplitude of the motion if the energy of the system is 12 J?

10.15 A mass is oscillating with amplitude A. (a) When the displacement is $x = \frac{1}{2}A$, what fraction of the energy is potential and what fraction is kinetic? (b) For what value of x in terms of A will the energy be half kinetic and half potential?

Answer: (a) $\frac{1}{4}$, $\frac{3}{4}$, (b) 0.707A.

10.16 A wooden block of mass 0.8 kg rests on a frictionless table connected to a spring ($k = 200$ N/m) as shown in Fig. 10-10. A 20-g bullet moving with a velocity $v = 500$ m/sec is shot into the block and remains embedded in it. What is the amplitude of the ensuing oscillatory motion?

Answer: 0.78 m.

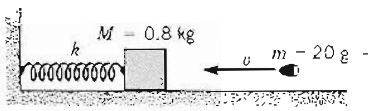


FIGURE 10-10 Problem 10.16.

10.17 A block of mass $m_1 = 3$ kg rests on a frictionless surface connected to a spring ($k = 150$ N/m). A second block of mass $m_2 = 1$ kg is launched toward m_1 with a velocity of 4 m/sec (see Fig. 10-11). After the collision, m_2 bounces back in the opposite direction with a velocity of 1 m/sec. (a) How much will the spring be compressed? (b) What fraction of the energy is lost in the collision?

Answer: (a) 0.24 m, (b) 0.42.

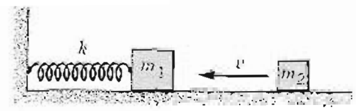


FIGURE 10-11 Problem 10.17.

10.18 A spring ($k = 200$ N/m) is compressed 10 cm between two blocks of mass $m_1 = 1.5$ kg and $m_2 = 4.5$ kg (see Fig. 10-12). The spring is not connected to the blocks, and the table is frictionless. What are the velocities of the blocks after they are released and lose contact with the spring? Assume that the spring falls straight down to the table.

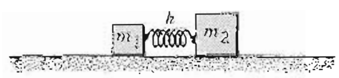


FIGURE 10-12 Problem 10.18.

10.19 A mass of 3 kg is connected to a spring of force constant 250 N/m on a horizontal surface. The coefficient of friction between the block and the surface is 0.1. The block is pulled 20 cm to the right and released. (a) How far to the left of the equilibrium point will the block move? (b) What is the total back and forth distance traveled by the block before it stops?

Answer: (a) 0.176 m, (b) 1.70 m.

11.1 INTRODUCTION

Waves are an important concept in physics. We can see water waves and readily demonstrate sound waves with elementary laboratory experiments. In 1801 Thomas Young showed that light can be considered wavelike by experimental analogies to the behavior of water waves. It will be shown in a later chapter that experiments with fundamental particles, such as electrons, demonstrate that they also have wave characteristics.

If we crack a whip, we produce a brief transverse displacement that can be seen to travel to the end of the whip. If we drop a pebble on the calm surface of a pond, a circular ripple is produced. This ripple travels away from the point where the pebble hit the water with constant speed and, as it reaches a given point of the water surface, it produces a temporary displacement of the water molecules. These are examples of *traveling* waves, which can transmit energy along a medium without any net translation of the particles in the medium through which the wave travels. These readily visible experiments with water or strings led early scientists to conclude that a wave is a disturbance that travels in a medium. When light was shown to have wave characteristics, even though it can travel in the vacuum of space that exists between the earth and the stars, an erroneous conclusion was drawn that there must exist a medium permeating the entire universe. This was called *aether*. As we will see in Chapter 16, because of the nature of the light wave, no medium is necessary for its propagation. Before we enter into the realm of modern physics, we must have a solid understanding of wave motion: the mathematical description of a wave, the parameters that characterize it, and the laws that govern its propagation. In the development of these ideas we will consider waves in a visible medium such as a string or a water surface.

11.2 WAVELENGTH, VELOCITY, FREQUENCY, AND AMPLITUDE

Suppose we are sitting in a boat on a body of water in which there is some wave motion, as in Fig. 11-1. If we measure the time between risings on the waves, we have a quantity known as the *period*, with symbol T, which is the time between successive risings. If instead, we ask how many risings did we experience per unit time, such as 1 h, this quantity is called the *frequency*, with symbol ν. Here ν = number of risings/unit time. Just as in the case of oscillatory motion considered in Chapter 10, the period and the frequency are reciprocals of each other.

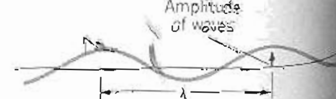


FIGURE 11-1

$$\nu = \frac{1}{T} \tag{11.1}$$

If we ask a friend in another boat to row away from us (in the direction of the wave motion) and to stop at the nearest location where he rises at the same instant that we do, the distance between us is called the *wavelength* of the wave with symbol λ. The speed of the wave through the water is the distance between the boats divided by the time it takes for our rising to reach him. This is called the *wave velocity*, with symbol υ, and

$$\upsilon = \frac{\lambda}{T}$$

or, from Eq. 11.1,

$$\upsilon = \lambda \nu \qquad (11.2)$$

This is a fundamental equation obeyed by all waves.

Another important parameter that characterizes a wave is its *amplitude*. The amplitude of a wave is the maximum value of the displacement it produces (see Fig. 11-1).

11.3 TRAVELING WAVES IN A STRING

Let us have a very long string stretched along the x direction. A pulsed displacement, such as plucking the string, introduced at one end of the string, will cause a transverse displacement of the string in the y or z direction. Let us for now consider only the transverse y direction because it is more easily drawn on a flat sheet of paper.

At $t = 0$, when the pulse is introduced, the string may look as in Fig. 11-2*a*. The wave pulse travels along the string and, consequently, the string will look differently some time later, see Fig. 11-2*b*. This clearly indicates that the transverse displacement of the string, y, varies with x—that is, the point of the string under consideration—and with time t. In mathematical terms we say that the wave pulse, y, is a function of x and t; that is, $y = f(x, t)$. The exact shape of the wave pulse, the precise form of the function $f(x, t)$, will be determined by the source producing the pulse and the nature of the string. One of the most important and most commonly found types of traveling waves is the sinusoidal traveling wave, a wave consisting of a series of consecutive sinusoidal pulses.

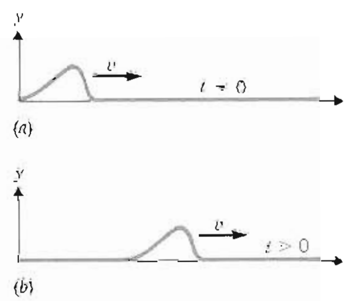


FIGURE 11-2 A transverse displacement in a string traveling in the positive x direction.

Let us attach the end of the string ($x = 0$) to a block connected to a spring hanging from the ceiling, as shown in Fig. 11-3. If we pull on the block, thus stretching the spring, and then release it, we know that the block will begin to oscillate. The y coordinate of the block, and therefore the transverse displacement of that end of the string, will be given by Eq. 10.9

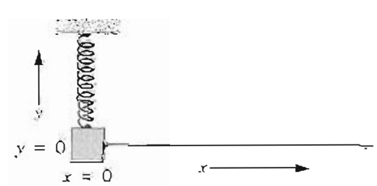


FIGURE 11-3 Arrangement for the production of sinusoidal traveling waves in a long string under tension.

$$y(x = 0, t) = A \sin(\omega t + \phi) \qquad (11.3)$$

This transverse displacement, introduced at $x = 0$, moves along the string and, as a result, sometime later other points on the string will begin to oscillate in the transverse y direction. If the velocity of the wave in the string is v, then the time it takes to travel a distance x along the string is x/v. Thus, at time t the displacement produced by the wave at a point x is the same as was the displacement at the origin ($x = 0$) at an earlier time $t - t_0$, where t_0 is the time that it takes the wave to reach x; that is, $t_0 = x/v$. Putting $t - t_0$ into Eq. 11.3 for t, we obtain

$$y(x,t) = A\sin[\omega(t - t_0) + \phi]$$

and on substituting $t_0 = x/v$ we have

$$y(x,t) = A\sin\left(\omega t - \frac{\omega}{v}x + \phi\right) \tag{11.4}$$

Eq. 11.4 is the general form of the wave equation, but for our purposes we do not require such generality. If we limit ourselves to waves such that $y = 0$ when both $x = 0$ and $t = 0$, then $\phi = 0$ or π. We can then write the commonly used versions of Eq 11.4 as

$$y(x,t) = A\sin(\omega t - kx) \quad \text{when } \phi = 0 \tag{11.5}$$

or

$$y(x,t) = A\sin(kx - \omega t) \quad \text{when } \phi = \pi \tag{11.5'}$$

where

$$k = \frac{\omega}{v} \tag{11.6}$$

Either of these versions of the travelling wave equation may be used, and the choice is a writer's preference.

The constant k that we have introduced is called the *propagation constant* (or *wave number*). (Note that this k is a new and different constant from the spring constant k introduced in Chapter 10.) It will be used extensively in the latter part of this book. Its physical significance will soon become apparent.

We can write an alternative representation of a sinusoidal traveling wave that differs from that of Eq. 11.5 in the direction of propagation of the wave that it represents. Equation 11.5 was derived by assuming that the wave traveled toward the right in the positive x direction. For a wave traveling toward the left,

$$y(x,t) = A\sin(kx + \omega t) \tag{11.7}$$

To show that Eq. 11.7 represents a wave traveling in the negative x direction, we look at a particular value of y, the wave displacement, and ask ourselves, as time t increases, what happens to x for that particular y value of the wave? To pick a particular, fixed value of y, the argument of the sine function in Eq. 11.7 must be kept constant, that is,

$$kx + \omega t = \text{constant} \tag{11.8}$$

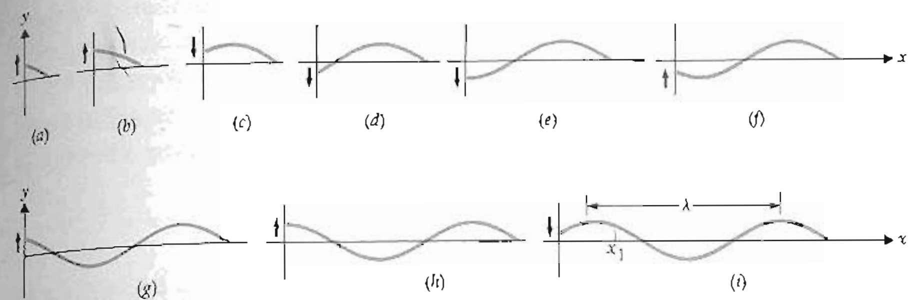


FIGURE 11-4 Stop-action diagrams showing the introduction of a traveling wave in a string by the experimental arrangement of Figure 11-3.

Obviously, as t increases, x must decrease if the left side of Eq. 11.8 is to remain constant. Therefore, the wave of Eq. 11.7 is moving toward decreasing values of x, that is, in the negative x direction.

To understand a traveling wave in a string produced by an oscillating source such as that of Fig. 11-3, let us examine some stop-action diagrams, such as those in Fig. 11-4. We see that the y displacement introduced by the oscillation travels to the right in the detailed sketches of Fig. 11-4*a–i*. More than one full wavelength is represented in Fig. 11-4*g–i*.

To gain insight into the physical significance of the wave represented by either Eq. 11.5′ or Eq. 11.7, let us analyze it from two different points of view.

Suppose we take a snapshot of the string as the wave travels through it. What will we see? Taking a snapshot of the string means setting t equal to an instantaneous value t_1 in Eq. 11.5′, which now becomes

$$y(x, t_1) = A\sin(kx - \theta_1) \tag{11.9}$$

where $\theta_1 = \omega t_1$ is a *phase shift* at t_1. This is shown best in Fig. 11-5 in comparison with a sine function plotted as a dashed line. In this figure the solid line is the snapshot at time t_1 and the dashed line is a sine curve with $y = 0$ at the origin. At the time of the photograph the solid line is phase shifted by an angle $\theta_1 = \omega t_1$ from the sine curve of $y = 0$ at $x = 0$. At some other time t_2 the snapshot would show a different phase shift θ_2. Even though there is a phase shift, the string looks like a sine wave. We can use Eq. 11.9 to find the wavelength λ of the wave. In Section 11-2, we defined λ as the separation between two risings in the body of water of Fig. 11-1. Similarly here, λ is the separation between two successive maxima in the transverse displacement of the string; for example, from Fig. 11-5

$$\lambda = x_2 - x_1$$

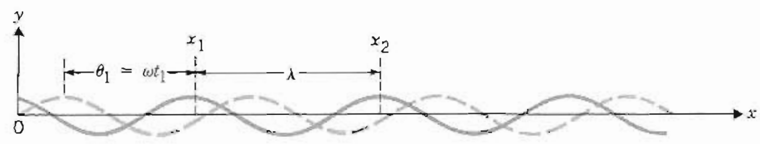


FIGURE 11-5

We identify x_1 and x_2 as two successive values of x for which the sine function in Eq. 11.9 equals +1; that is,

$$kx_1 - \theta_1 = \frac{\pi}{2}\,\text{rad} \tag{11.10}$$

and

$$kx_2 - \theta_1 = \left(\frac{\pi}{2} + 2\pi\right)\text{rad} \tag{11.11}$$

Solving Eqs. 11.10 and 11.11 simultaneously, we obtain

$$kx_2 = kx_1 = 2\pi$$

$$x_2 - x_1 = \frac{2\pi}{k}$$

$$\lambda = \frac{2\pi}{k} \tag{11.12}$$

We see, therefore, that the propagation constant k in Eq. 11.5′ yields, through the relation of Eq. 11.12, the wavelength of the wave.

An alternative way of looking at Eq. 11.5′ is to consider a particular point in the string of Fig. 11-4*i* and to analyze the motion of that point as a function of time. We can place a little light bulb at that point in the string and follow the motion of the bulb. Suppose we choose the particular point in the string as x_1 in Fig. 11-4*i*. This means we set x equal to a constant value x_1 in Eq. 11.5′, which now becomes

$$y(x_1, t) = A\sin(\theta_1' - \omega t) \tag{11.13}$$

where $\theta_1' = kx_1$ is a constant phase shift that depends on the point chosen in contrast to the previous analysis, which showed that the phase shift depended on the time of the snapshot. We immediately recognize Eq. 11.13 as being similar to Eq. 10.9, which described the oscillatory motion of the body attached to a spring. With the position x fixed, y will vary with $\sin \omega t$ and the little bulb will undergo simple harmonic motion with amplitude A and frequency

$$\nu = \frac{\omega}{2\pi} \tag{11.14}$$

It should be noted that Eq. 11.13, with a change in x and therefore with the corresponding phase shift θ_1', describes the motion of any other point in the string. As the wave moves through the string, all the particles in the string oscillate with the same amplitude and frequency, although *out of phase* with one another, that is, with different phase shifts.

We conclude this section with an alternative demonstration of the relation between frequency and wavelength, Eq. 11.2. Combining Eq. 11.14 and Eq. 11.12, we have the product

$$\lambda\nu = \frac{2\pi}{k}\frac{\omega}{2\pi} = \frac{\omega}{k}$$

By definition $k = \frac{\omega}{v}$ (Eq. 11.6), therefore

$$\lambda\nu = \frac{\omega}{k} = \frac{\omega}{\frac{\omega}{v}} = v \tag{11.2}$$

which is the result found earlier. This result is valid for all waves whether in a medium such as a string, or water or air, or in a vacuum such as light waves.

EXAMPLE 11-1

A mass of 0.2 kg suspended from a spring of force constant 1000 N/m is attached to a long string as shown in Fig. 11-3. The mass is set into vertical oscillation, and the distance between successive crests of the waves in the string is measured to be 12 cm. What is the velocity of waves in the string?

Solution We use Eq. 10.12 to find the frequency of oscillation

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000\,\text{N/m}}{0.2\,\text{kg}}} = 70.71\,\text{rad/sec}$$

$$\nu = \frac{\omega}{2\pi} = \frac{70.71\,\text{rad/sec}}{6.28} = 11.26\,\text{Hz}$$

The velocity of waves in a medium can be found with Eq. 11.2

$$\begin{aligned} v &= \lambda\nu \\ &= 12 \times 10^{-2}\,\text{m} \times 11.26\,\text{sec}^{-1} \\ &= 1.35\,\text{m/sec} \end{aligned}$$

11.4 ENERGY TRANSFER OF A WAVE

One of the most important aspects of wave motion is that it provides a mechanism for the transfer of energy. A particle in a string before the wave arrives has no mechanical energy. If a sinusoidal wave arrives at the location of the particle, the particle begins to execute simple harmonic motion, and it therefore acquires kinetic as well as potential energy. *The wave has given energy to the particle because the wave carries energy with it.*

In fact if we think carefully, wave motion is one of the two general mechanisms available to transport energy from one point to another. The other occurs when one or more particles move from one point to another and in so doing bring their kinetic energy with them. This kinetic energy can be transferred to other particles in the medium through which they propagate. There are, however, two obvious differences between these two mechanisms; one of them will be crucial in the development of quantum mechanics in a later chapter.

1 The first difference is that waves transfer energy without transfer of matter, unlike the motion of particles.
2 The second is that the energy of a beam of particles is localized (it is where the particles are at a given instant). In a wave the energy is distributed over the entire space occupied at a given instant by the wave. (When the ripples in the pond move outward from their source, all the water in the region of the wave is displaced.)

We will now calculate the rate—that is, energy per unit time—at which energy is transmitted by a wave in a string, noting that a similar calculation, leading to similar results, may be made for any other type of wave. Let us consider the sinusoidal traveling wave represented by Eq. 11.5′.

$$y = A\sin(kx - \omega t) \qquad (11.5')$$

The rate of energy production, consumption, or transmission was defined in Eq. 5.15 as power P. We can obtain P by calculating the energy crossing a given point in a string in 1 sec, for example, point D in Fig. 11-6. This will be equal to the wave energy of the string particles in one wavelength multiplied by the number of wavelengths passing point D in 1 sec, that is, by the frequency ν.

$$P = (\text{energy per wavelength}) \times \nu \qquad (11.15)$$

To find the energy in one wavelength we note, as shown in the previous section, that each particle in the string is oscillating with the same amplitude A. Because the total energy of an oscillating particle is proportional to the square of the amplitude of oscillation (see Eq. 10.26 et seq.), we conclude that all the particles in the vibrating string have the same energy. At any given time, the energy of a particular particle may be all kinetic or all potential or a mixture. In Fig. 11-6, the energy of particle C is all kinetic, because C is passing through the equilibrium point. On the other hand, the energy of particle B is all potential, because it is about to reverse the direction of its transverse motion and its velocity is zero.

To obtain the kinetic energy of the particles in the string we need an expression for the transverse velocity v_y. This can be obtained by differentiating y in Eq. 11.5′ with respect to time. Because x is also a variable in the expression for y, we will indicate

The splashing of the water caused by the sound waves of a tuning fork is a vivid illustration of the transfer of energy by a wave.

its omission in the derivative by writing v_y as a partial derivative with respect to t; that is,

$$v_y = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t} A \sin(kx - \omega t)$$

$$v_y = -\omega A \cos(kx - \omega t) \tag{11.16}$$

We can now calculate the energy of particle C of mass Δm.

$$E(\text{for particle } C) = \frac{1}{2} \Delta m v_{y\,\text{max}}^2 \tag{11.17}$$

But from Eq. 11.16 $v_{y\,\text{max}} = -\omega A$, because the maximum value of the cosine function is 1. Eq. 11.17 becomes

$$E(\text{for particle } C) = \frac{1}{2} \Delta m \omega^2 A^2 \tag{11.18}$$

Because the energy of all the particles is the same, we can obtain the energy in one wavelength by replacing Δm in Eq. 11.18 with the mass contained in one wavelength. If the amplitude of the wave is small compared with the wavelength (a situation often satisfied), then the mass in one wavelength is $\mu\lambda$, where μ is the mass per unit length of the string. Therefore,

$$\text{Energy per wavelength} = \frac{1}{2} \mu \lambda \omega^2 A^2 \tag{11.19}$$

Substituting Eq. 11.19 for the energy per wavelength in Eq. 11.15, we obtain

$$P = \frac{1}{2} \mu \lambda \nu \omega^2 A^2$$

or

$$P = 2\pi^2 \mu v \nu^2 A^2 \tag{11.20}$$

where we have made use of the fact that $\lambda\nu = v$ and $\omega = 2\pi\nu$.

Although Eq. 11.20 has been derived for a wave in a string, two important features hold for any other type of wave: The *power transported by a wave is proportional to the square of the amplitude* and to the velocity of propagation of the wave. We will use these important results later in our discussions of the principles of modern physics.

When considering waves that propagate in three dimensions, such as sound waves or light waves, it is convenient to talk about the energy flowing through a given area of the medium traversed by the wave. The unique term *intensity*, with symbol I, is used for this purpose. The intensity is defined as the power transmitted per unit area perpendicular to the direction of propagation of the wave. Clearly, intensity and power are related by a simple geometric factor. Thus, the intensity of the wave is also proportional to the square of the amplitude. In the SI system, intensity has units of W/m^2.

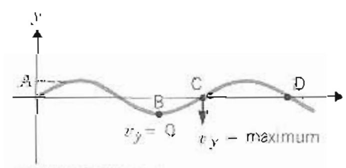


FIGURE 11-6

PROBLEMS

11.1 The speed of sound in air is about 330 m/sec, whereas the velocity of light is 3×10^8 m/sec. If you see a flash of lightning and count 8 sec before you hear the thunder, how far away was the lightning?

Answer: 2.64×10^3 m.

11.2 If the principal audio frequency of a thunderclap is the lowest the ear can hear, about 20 Hz, what is the wavelength of the sound wave? The speed of sound in air is about 330 m/sec.

11.3 The speed of all electromagnetic waves in air, both visible and invisible, is 3×10^8 m/sec. The AM radio band ranges in frequency from 550 to 1600 kHz. What is the range of wavelengths? The FM band ranges from 88 to 108 MHz ($1\ \text{MHz} = 10^6$ Hz). What is its range of wavelengths?

11.4 The range of sound frequencies detectable by the human ear is 20 to 20,000 Hz. What is the range of wavelengths? The velocity of sound in air is 330 m/sec.

11.5 A rule of thumb for finding the distance where a flash of lightning occurs is to count the number of seconds from the moment one sees the lightning to the moment one hears the thunder. The distance in kilometers is the number of seconds divided by 3. How accurate is this rule? (See problem 11-1.)

11.6 In an experiment designed to measure the velocity of sound waves in copper, a blow is struck at one end of a copper rod. Detectors at the other end measure the time interval between the arrival of the sound pulse through the rod and the arrival of the sound pulse through the air. If the rod is 3 m long and the sound pulse that traveled through the rod arrives 8.01×10^{-3} sec earlier than the sound pulse that traveled through the air, what is the velocity of sound in copper? The velocity of sound in air is 330 m/sec.

Answer: 2775 m/sec.

11.7 In the wave configuration shown in Fig. 11-7 the length of the string is 4.2 m, the frequency of the wave is 1.2 Hz, and the amplitude is 0.05 m. What is the speed of the wave? Note that $\lambda \gg$ amplitude.

Answer: 2.52 m/sec.

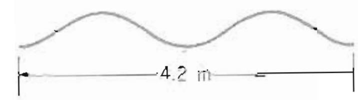


FIGURE 11-7 Problem 11.7.

11.8 The equation of a transverse wave traveling along a very long string is given by

$$y = 6.0 \sin(0.020\pi x + 4.0\pi t)$$

where x and y are expressed in centimeters and t in seconds. Find (a) the amplitude, (b) the wavelength, (c) the frequency, (d) the speed of propagation, (e) the direction of propagation of the wave, and (f) the maximum transverse speed of a particle in the string.

11.9 The equation of a traveling wave in a long stretched string is $y = 10^2 \sin(32t - 4x)$m, where x is in meters and t is in seconds. What is the velocity of the wave in the string?

11.10 Write the equation of motion of a traveling wave for the string in problem 11-7 using the same amplitude. Assume that the wave travels in the positive x direction.

11.11 Write the equation for a wave traveling in the negative direction along the x axis and having an amplitude of 0.010 m, a frequency 550 Hz, and a speed 330 m/sec.

Answer: $y = 0.010 \sin(3.33\pi x + 1100\pi t)$ m.

11.12 Consider the situation illustrated in Fig. 11-3. Let the spring constant $k = 200$ N/m and the mass of the block $m = 2$ kg. The block is given an upward initial kick to start it oscillating. Sketch the shape of the string at $t = 0.15$ sec, $t = 0.3$ sec, $t = 0.45$ sec, $t = 0.6$ sec, and $t = 1.2$ sec.

11.13 A triangular pulse of height 0.5 m and length 2 m moves along the positive x direction on a string with velocity 12 m/sec (see Fig. 11-8). At $t = 0$ the pulse is between $x_1 = 1$ m and $x_2 = 3$ m. Plot the transverse velocity of point x_2 as a function of time.

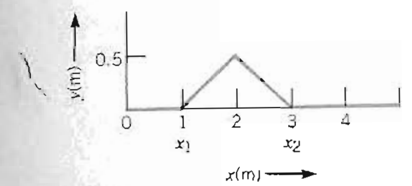


FIGURE 11-8 Problem 11.13.

11.14 A sinusoidal traveling wave on a string has a frequency $\nu = 15$ Hz and a velocity $v = 7.5$ m/sec. (a) How far apart are two points whose transverse displacements are phase-shifted by 30°? (b) At a particular point on the string, what is the phase difference between two displacements occurring 0.05 sec apart?

Answer: (a) 4.17×10^{-2} m, (b) 270°.

11.15 A block connected to a rigid rod is raised from some initial position $y = 0$ with constant velocity $v_y = 20$ m/sec for 0.4 sec. The block is then suddenly lowered to its initial position and then raised again with the same velocity for the same amount of time. The cycle is repeated indefinitely. A long string under tension is attached to the side of the block (see Fig. 11-9). Let the wave velocity in the string be 5 m/sec. (a) Sketch the shape of the string, using approximate dimensions for the x and y coordinates, at $t = 0.2$ sec, $t = 0.4$ sec and $t = 1.2$ sec. (b) What is the spacing of the pulses on the string?

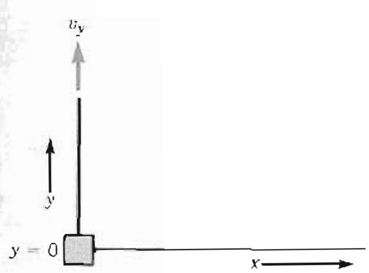


FIGURE 11-9 Problem 11.15.

11.16 A laser produces light pulses of energy 5 J and duration 2×10^{-9} sec. The width of the beam is 1 mm^2. What is the intensity of the laser light?

11.17 A sinusoidal traveling wave of amplitude 2 cm and wavelength 50 cm moves along a string with velocity $v = 6$ m/sec. (a) What is the maximum transverse velocity of the particles in the string? (b) What is the maximum transverse acceleration of the particles in the string?

Answer: (a) 1.51 m/sec, (b) 113.7 m/sec^2.

11.18 If the string of problem 11.17 is 30 m long and has a mass of 6 kg, what is the power transmitted by the wave?

11.19 Suppose that during the transmission of waves through a string the frequency is suddenly doubled while maintaining the same amplitude and velocity of propagation, what will happen to the magnitude of the power transmitted? Suppose that the amplitude is doubled while holding the frequency fixed, what will happen to the power?

11.20 Consider a point source emitting waves in all directions. If the medium through which the waves propagate is isotropic, the velocity of propagation will be the same in all directions. As a result, points in the medium where the wave has a certain phase are equidistant from the source and therefore lie on a spherical shell with the source at the center. These waves are called spherical waves. Consider waves emitted from a 5-W source in a nonabsorbing medium. (a) What is the intensity of the waves 1 m away from the source? (b) What should the power of the source be in order that the intensity of the waves 1 m away be the same as that of the laser in problem 11.16?

11.21 A point source emits spherical waves in a nonabsorbing medium (see problem 11.20.) The intensity at some unknown distance from the source is 25 W/m^2. The intensity at some point 10 m farther away from the source is 16 W/m^2. (a) How far is the source from the first point? (b) What is the power output of the source?

Answer: (a) 40 m, (b) 503 kW.

12.1 INTRODUCTION

In the preceding chapter we introduced the concept of waves as a periodic disturbance of a medium. We used familiar concepts such as waves on a string or in water to illustrate the phenomena. The general equation for a traveling wave was derived as was the concept of phase shift. In this chapter we will start with the behavior of two waves when they come together and the effect produced by their relative phase. We will first discuss this phenomenon with waves in water and then extend it to light waves. At this point we will assume, as did early investigators, that air could be the substance in which light-wave motion occurs. However, we will show in a later chapter that light waves do not require some substance or medium to support them.

12.2 THE SUPERPOSITION PRINCIPLE

One of the fundamental principles governing the propagation of waves is called the *superposition principle*. What happens when two different waves meet? Experiments show that waves can move through the same region of space independently and, as a result, when they meet the resultant wave is simply the algebraic sum of the individual waves. (The superposition principle does not hold for waves of very large amplitude in deformable media.) Figure 12-1 shows what happens when a square pulse in the string meets a triangular pulse moving in the opposite direction. In Fig. 12-1*a* two positive waves approach each other. The resulting displacement of the string is the addition of the displacements that each pulse would have produced in the absence of the other (Fig. 12-1*b*). After meeting, the waves move on unaffected (Fig. 12-1*c*). If one wave pulse is positive and the other is negative, as in Fig. 12-1*d*–*f*, the negatively directed pulse subtracts from the positive pulse. These two examples show that the resultant pulse while the two waves are passing one another is the algebraic sum of the two waves.

The superposition principle leads to a wave phenomenon known as *interference*. Suppose two waves with the same wavelength, velocity, and amplitude, but from different sources, travel together in the same direction. What will be the amplitude of the resulting wave? Figure 12-2*a* shows that if they are in phase, the total amplitude at any point will be the simple sum of the two. If they are out of phase by one-half wavelength, the resulting amplitude will be zero (see Fig. 12-2*b*). The first case is called *constructive interference*, and the second is called *destructive interference*. It is important to note that if either wave is shifted to the right or to the left by a whole wavelength, the situation is unchanged. However, if the shift is by only a half wavelength the situation is reversed; that is, constructive interference becomes destructive and vice versa.

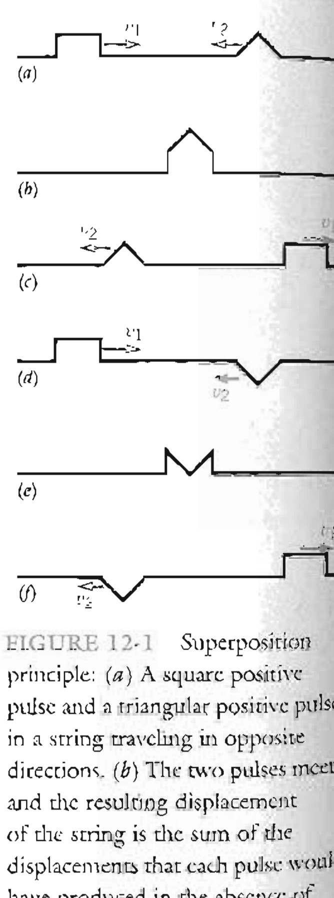


FIGURE 12-1 Superposition principle: (*a*) A square positive pulse and a triangular positive pulse in a string traveling in opposite directions. (*b*) The two pulses meet, and the resulting displacement of the string is the sum of the displacements that each pulse would have produced in the absence of the other. (*c*) After meeting, the pulses move away unaffected. (*d*) A positive square pulse and a negative triangular pulse moving in opposite directions. (*e*) The pulses meet, the resulting displacement is the difference (algebraic sum) of the two pulses. (*f*) After meeting, the two pulses move away unaffected.

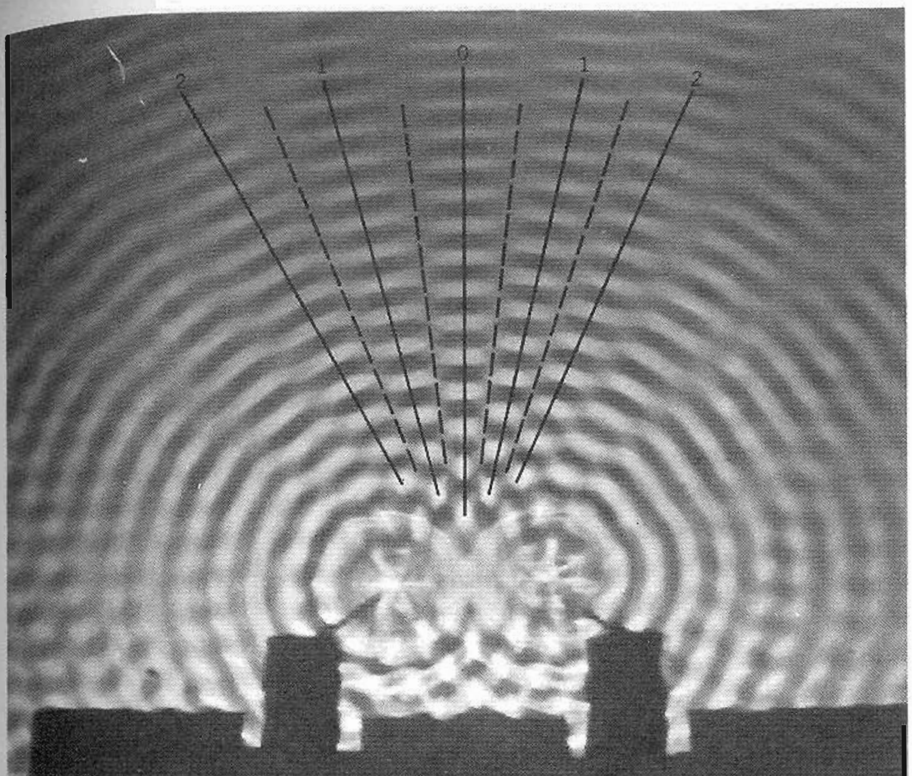


FIGURE 12-3 Ripple-tank demonstration of the phenomenon of interference. Two vibrators striking the water surface in the tank at periodic intervals produce two circular wave patterns. As the two wave patterns cross each other, an interference pattern results. Along the solid lines the waves from the two sources interfere constructively, that is, the displacement of the water is large. Along the dashed lines the interference is destructive, and the displacement of the water is zero. There are additional paths of constructive and destructive interference that have not been marked with lines.

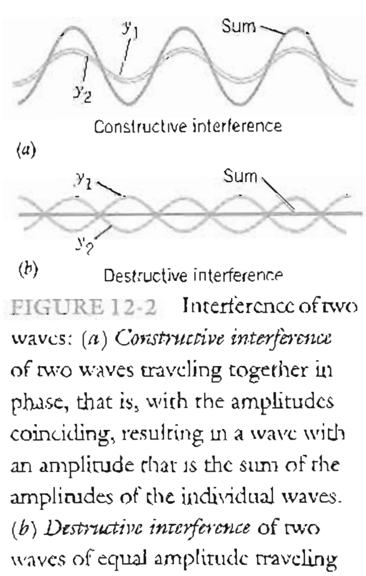


FIGURE 12-2 Interference of two waves: (*a*) *Constructive interference* of two waves traveling together in phase, that is, with the amplitudes coinciding, resulting in a wave with an amplitude that is the sum of the amplitudes of the individual waves. (*b*) *Destructive interference* of two waves of equal amplitude traveling together with a phase difference of one-half wavelength resulting in a wave of zero amplitude.

12.3 INTERFERENCE FROM TWO SOURCES

If pebbles or water drops fall at regular intervals in still water, a pattern of circular waves, each constantly increasing in radius, will be established. If a similar situation with the same frequency of disturbance occurs nearby, the circular traveling waves will cross one another, producing an interference pattern. Note that the restriction of the previous section that the waves travel together in the same direction has now been removed. At some points the interference will be constructive, and at others it will be destructive. A simple laboratory demonstration of this is shown in Fig. 12-3. This is a photograph of what is called a *ripple tank*. This is a tray of water illuminated from below. Instead of having drops of water falling, two vibrators are placed in the tray and the frequency and amplitude of vibration can be controlled more accurately than can that of falling drops of water. The centers of the circles at the bottom are the locations at which the vibrators are located. Some of the paths along which constructive interference occurs are indicated with solid lines. The dashed lines between the paths labeled 0 and 1 are regions of destructive interference.

Figure 12-4 is a schematic representation of the photograph in Fig. 12-3 in which the wave crests, represented by the circular lines, proceed outward from the two sources S_1 and S_2. It is seen in this figure that the phenomenon of Fig. 12-3 is a purely geometric one that can be reproduced on paper with a compass and ruler. The distance between the crests is the wavelength λ. The troughs are located halfway between the crests. As in the photograph, the solid lines labeled 0, 1, 2, . . . represent the paths of constructive interference.

We notice in this figure that the paths of constructive interference are symmetric about the line labeled 0. Therefore, we need to consider only one group either above or below the 0 line. We will consider the ones above, knowing that the results will be the same for the ones below. Constructive interference occurs along these paths because the crests from the two sources coincide and add to the disturbance of the water. This is the criterion presented in the previous section for constructive interference. Along the path labeled 0 in Fig. 12-4, the first crest from S_1 coincides with the first crest from S_2, the second crest from S_1 with the second from S_2, and so on. Along the path labeled 1, the first crest from S_2 coincides with the second crest from S_1, the second from S_2 with the third from S_1, and so on. We may view these waves as having traveled for some distance from their sources along their respective paths. When two waves travel in the same medium, the difference in the distances traveled by them from their respective sources to a common point is called the *path difference*. Keeping in mind that the separation between successive crests is the wavelength λ, we can now state the criterion for constructive interference as follows: When waves from two sources are emitted in phase, ***constructive interference occurs when the path difference is zero, or one wavelength, or an integral multiple of wavelengths $n\lambda$***. This can be formally shown with the trigonometric relation for the sum of sines,

$$\sin a + \sin b = 2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) \qquad (12.1)$$

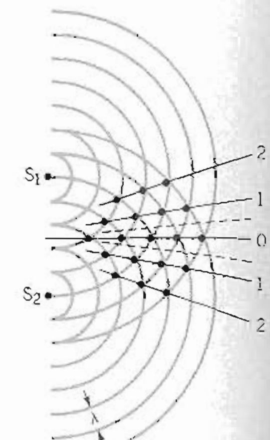


FIGURE 12-4 Geometric representation of the photograph in Fig. 12-3. The crests of the water waves are represented by circular lines whose centers coincide with the location of the vibrators. The distance between adjacent crests is the wavelength. The troughs are halfway between the circular lines. The crests from the two sources coincide along the solid lines labeled 0, 1, 2, where constructive interference occurs. Along the dashed lines, the crests from one source coincide with the troughs from the other, resulting in paths of destructive interference.

Let us consider a point P whose distances from S_1 and S_2 are x_1 and x_2, respectively, as shown in Fig. 12-5. From Eq. 11.5′, the wave y_1 from S_1 and the wave y_2 from S_2 at P are

$$y_1 = A \sin(kx_1 - \omega t)$$

$$y_2 = A \sin(kx_2 - \omega t)$$

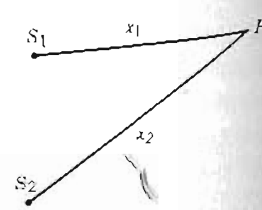


FIGURE 12-5 Arbitrary point P in Fig. 12-3 or 12-4.

From the superposition principle, the resulting wave will be

$$\begin{aligned} y &= y_1 + y_2 \\ &= A[\sin(kx_1 - \omega t) + \sin(kx_2 - \omega t)] \end{aligned} \qquad (12.2)$$

Now let the path difference be an integral multiple of the wavelength, that is,

$$x_2 - x_1 = n\lambda, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

and

$$x_2 = x_1 + n\lambda$$

or, because from Eq. 11.12

$$\lambda = \frac{2\pi}{k}$$

$$x_2 = x_1 + \frac{2\pi n}{k},$$

then Eq. 12.2 becomes

$$y = A[\sin(kx_1 - \omega t) + \sin(kx_1 + 2\pi n - \omega t)] \tag{12.3}$$

Using Eq. 12.1 gives

$$y = 2A \sin \frac{1}{2}(kx_1 - \omega t + kx_1 - \omega t + 2\pi n) \cos \frac{1}{2}(2\pi n)$$

$$y = 2A \sin(kx_1 - \omega t) \tag{12.4}$$

In the last step we let $kx_1 - \omega t = \theta$ and used the fact that $\sin(\theta + n\pi) = -sin\theta$ and $\cos n\pi = -1$ if n is an odd integer, and $\sin(\theta + n\pi) = \sin\theta$ and $\cos n\pi = 1$ if n is an even integer. Equation 12.4 formally verifies that when the path difference is an integral multiple of the wavelength, the resulting wave has an amplitude that is twice that of y_1 or y_2 at that point.

Returning to Fig. 12-4, we note that the dashed lines, representing paths where destructive interference occurs, correspond to points whose distances from S_1 and S_2 differ by $1/2\ \lambda$, or (from Eq. 11.12) $x_2 - x_1 = \lambda/2 = \pi/k$. The resulting wave in this case will be

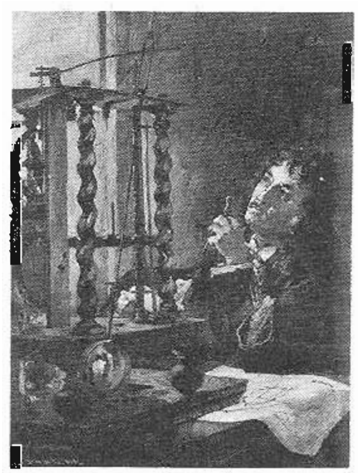
$$y = y_1 + y_2$$

$$= A[\sin(kx_1 - \omega t) + \sin(kx_1 + \pi - \omega t)]$$

Using Eq. 12.1, we obtain

$$y = 2A \sin \frac{1}{2}(kx_1 - \omega t + kx_1 - \omega t + \pi) \cos \frac{1}{2}\pi$$

But $\cos \pi/2 = 0$, and therefore $y = y_1 + y_2 = 0$; this by definition is destructive interference. The same result is obtained if $x_2 - x_1 = \frac{3}{2}\lambda$ or $\frac{5}{2}\lambda$ and so on.



Christian Huygens (1627–1675)

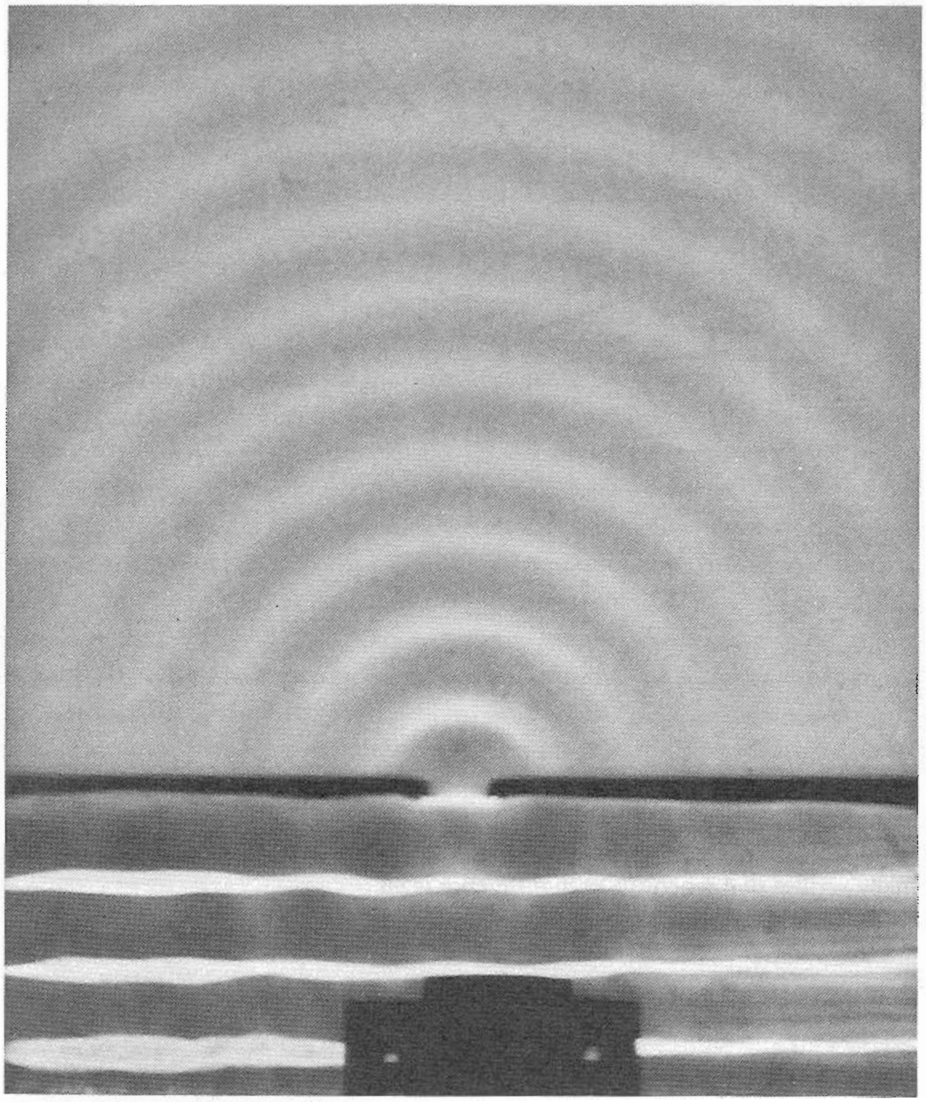


FIGURE 12-6 Huygens' principle. Parallel wave fronts in the ripple tank strike a barrier with a small opening. The opening becomes a source of circular waves.

12.4 DOUBLE SLIT INTERFERENCE OF LIGHT

If a series of either plane waves or large radius spherical waves strike a barrier with a small opening, circular waves are propagated beyond the opening as if the opening were a point source. The enlarging circumference of these waves is called a *wave front*. Figure 12-6 illustrates this propagation with water waves in a still tank. This phenomenon illustrates what is known as after the Dutch physicist, Christian Huygens (1629–1695). It is more generally stated that every point on a wave front can be considered as a source of secondary wavelets that spread out in all directions with a speed and wavelength equal to those of the propagating wave. The newly propagated wave front is the tangent to these secondary wavelets. Thus, the spherical wavelet passing through the opening in the barrier of Fig. 12-6 produces a secondary wave front that itself produces other wavelets as shown in Fig. 12-7.

Huygens demonstrated that the known facts about the propagation of light could be explained by using his principle. It was many years, however, before light was accepted as a wave phenomenon. This came about when the English physician Thomas Young (1773–1829) performed the first successful experiment that exhibited the interference of light in 1801. The nature of light is discussed in more detail in Chapter 16. For now, we mention that visible light has a wavelength that ranges from about 4×10^{-7} m to 7×10^{-7} m, where the lower values appear to us as violet and the higher values as red. A unit of length often used in specifying the wavelength of light is the Ångstrom, with symbol Å; $1\,\text{Å} = 10^{-8}$ cm $= 10^{-10}$ m, and therefore the wavelength of visible light ranges from about 4000 Å to 7000 Å.

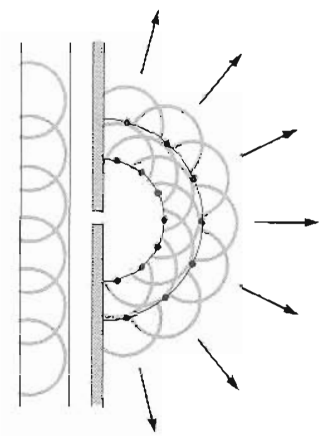


FIGURE 12-7 Geometric representation of the phenomenon in Fig. 12-6. The opening in the barrier produces a circular wave front. Each point in that wave front produces secondary circular wavelets, and the new propagated wave front is the tangent to these secondary wavelets.

Figure 12-8*a* illustrates a schematic arrangement, similar to the one used by Young, to determine the phenomenon of interference with light. A monochromatic light source (a single wavelength) shines on an opaque screen with two narrow slits S_1 and S_2. According to Huygens' principle, these two slits become point sources of light. If we let the light that passes through the slits fall on a screen, we will observe a pattern of bright and dark lines that indicates constructive and destructive interference. A plot of the light intensity on the screen is schematically represented in Fig. 12-8*a*. Figure 12-8*b* is a photograph of the interference pattern observed on the screen.

We can use the principles developed in the previous section to find expressions for the position of the interference maxima and minima. Figure 12-9*a* shows a geometric construct of lines drawn from each of the two slits to a point P of constructive interference. As will become evident soon, for the interference pattern to be easily observable, the separation between the slits, d, cannot be much greater than the wavelength. This implies that in the case of light d might be a few microns (10^{-6} m) whereas the distance, D, between the slits and the screen could be several centimeters, that is, $D \gg d$. We therefore conclude that the two lines S_1P and S_2P are almost parallel. The situation of Fig. 12-9*a* can thus be approximated by that shown in Fig. 12-9*b*. In Fig. 12-9*b*, a perpendicular has been dropped from slit S_2 to point Q on the line S_1P. Angles indicated by θ are equal because their sides are mutually perpendicular. The extra distance S_1Q traveled by the wave from slit S_1 is the path difference, with symbol δ, between the two waves when they arrive at P. It is seen from triangle S_1S_2Q that

Thomas Young (1773–1829).

$$\delta = d \sin\theta \qquad (12.5)$$

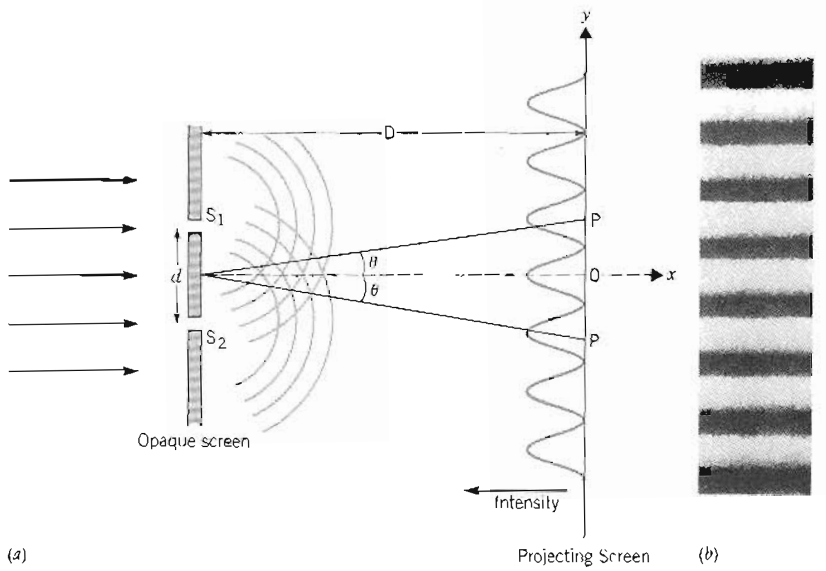


FIGURE 12-8 (*a*) Schematic of Young's double slit interference of light experiment. Light passing through two small slits S_1 and S_2 in an opaque screen produce an interference pattern on a projecting screen to the right of the slits. (*b*) Photographof the interference pattern on the projecting screen showing a pattern of and dark fringes. (*Source*: Cagnet et al., *Atlas of Optical Phenomena*, Springer-Verlag, New York, 1971.)

As we have shown in the preceding section, constructive interference at point P can occur only if this path difference is an integral multiple of wavelengths $n\lambda$. The condition for *constructive interference* in this case becomes

$$d \sin \theta = n\lambda, \quad \text{where } n = 0, 1, 2, 3, \ldots \qquad (12.6)$$

Similarly, destructive interference will occur when the path difference is $1/2\,\lambda$, $3/2\,\lambda$, $5/2\,\lambda$, or in general $[(2n+1)/2]\,\lambda$, where $n = 0, 1, 2, 3, \ldots$. The condition for

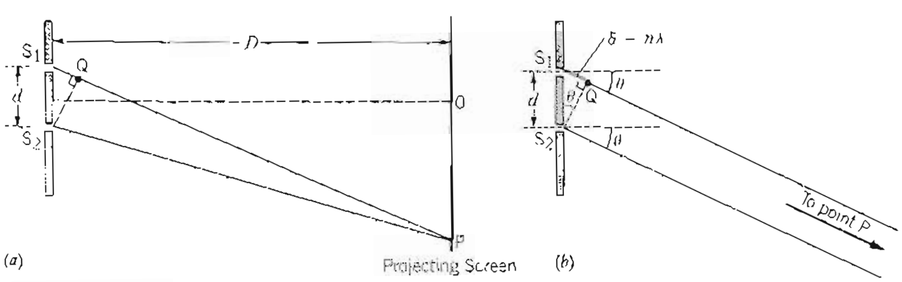


FIGURE 12-9 (*a*) Geometric construct of lines from each of the slits in Fig. 12-8 to a point P on the projecting screen where constructive interference is observed. (*b*) When $D \gg d$, the two lines are approximately parallel. The difference between the two paths traveled by the light from S_1 and S_2 is $\delta = n\lambda$.

destructive interference can be stated as

$$d \sin \theta = \frac{2n+1}{2}\lambda, \quad \text{where } n = 0, 1, 2, 3, \ldots \qquad (12.7)$$

We recall that as θ increases, $\sin \theta$ also increases. We conclude from Eq. 12.6 that for a constant λ, the angular separation between successive interference maxima decreases with increasing spacing between the slits, d. A simple laboratory demonstration of this is shown in Fig. 12-10. These are photographs of a ripple tank like the one used in the demonstration of Fig. 12-3. A comparison of Fig. 12-10*a* and 12-10*b* shows that if λ is kept constant, the angular separation between interference maxima decreases

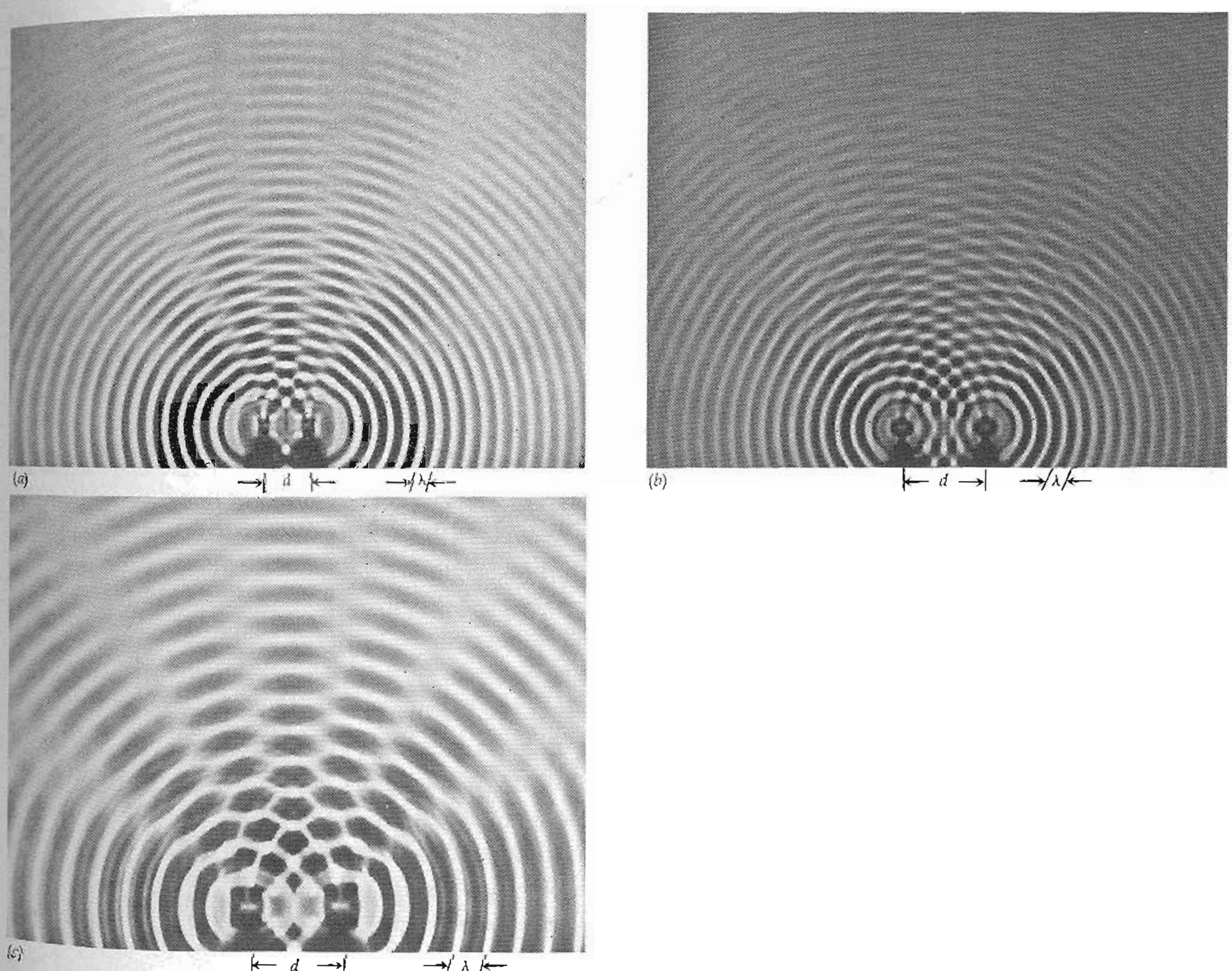


FIGURE (12-10) Interference patterns in the ripple tank of Fig. 12-3. (*a*) and (*b*) If λ is kept constant, the angular separation between the interference maxima decreases when the separation d between the sources is increased. (*a*) and (*c*) Both d and λ are doubled, the angular separation between interference maxima remains the same.

as the separation between the sources increases. Similarly a comparison of Fig. 12-10*a* and 12-10*c* indicates that when the separation between the sources is doubled, the wavelength must also be doubled to have the same angular separation. It should also be clear from Eq. 12.6 that if $d \gg \lambda$, a large number of interference maxima will occur within a small angle and, as a result, the pattern will be difficult to observe. To illustrate the point, let us assume that $d = 1000\,\lambda$. If we consider a small angle θ, for example, $\theta = 1°$, we can solve for n in Eq. 12.6, and we will get

$$1000\lambda \sin 1° = n\lambda$$

therefore

$$n = 1000 \sin 1° = 1000 \times 0.017 = 17$$

Seventeen intensity maxima will be formed within an angle of 1°. It is clear that unless the distance from the slits to the screen is very large, the interference pattern will not be observable.

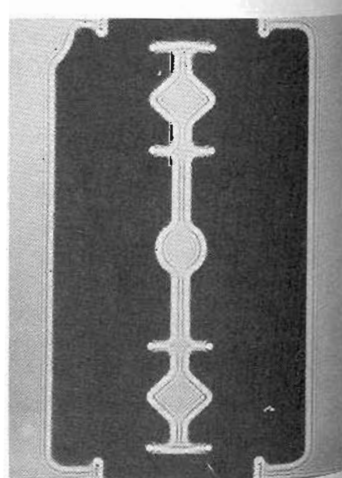
Although we will not derive the relation, it should be mentioned at this point that a derivation similar to that of the two-slits can be made for multiple slits of the same width and the same separation distance: Eqs. 12.6 and 12.7 also apply for multiple slits. An opaque piece of glass with multiple slits is called a ***diffraction grating***.

12.5 SINGLE SLIT DIFFRACTION

In the preceding discussion of double slit interference of light, we assumed that the two openings in the opaque screen acted as point sources. We will see that this assumption is correct if the size of the openings is smaller than the wavelength. If the opening is greater than the wavelength but of a size comparable to a few wavelengths, then light waves passing through different portions of a single slit will interfere with each other giving rise to a phenomenon known as ***single slit diffraction***.

The method used in Section 12.4 can be employed to analyze the diffraction of light by a single slit. Figure 12.11 illustrates schematically the passage of individual wavelets through a single slit of width d. The pattern seen on a screen to the right of the slit appears with a central bright maximum with alternating bright and dark fringes on either side, with the bright fringes diminishing in intensity as the angle from the normal increases. This is illustrated in Figs. 12-12*a* and *b*.

In our treatment, we will assume that the size of the slit is much smaller than the distance from the slit to the screen. Under this condition, the lines of propagation of the waves emanating from different points in the slit are approximately parallel. In Figs. 12-11*a* and 12-12*a* all forward-directed waves strike the screen in phase, because the path difference is zero. These waves give rise to the central maximum. In Fig. 12-11*b*, wave A has a path difference of $\lambda/2$ from wave B and interferes destructively. This



Photograph illustrating the diffraction of light by the fine edges of a razor blade.

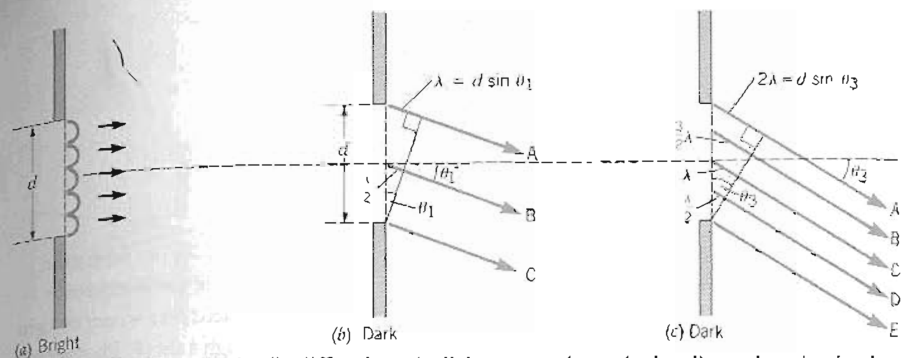


FIGURE 12-11 Single slit diffraction. As light passes through the slit, each point in the slit becomes a secondary source of light. (*a*) All forward-directed waves arrive at the screen in phase, giving rise to the central maximum of Fig. 12-12. (*b*) Waves from the upper half of the slit interfere destructively with those from the lower half, giving rise the first diffraction minimum in Fig. 12-12. (*c*) Waves between A and B interfere destructively with waves between B and C, those between C and D also interfere destructively with those between D and E. This results in the second interference minimum in Fig. 12-12.

destructive interference effect occurs across the entire slit because any wave slightly below A will interfere destructively with the wave at the same distance below B and so on. Therefore, a dark band will appear on the screen for angle θ_1, where

$$\sin\theta_1 = \frac{\lambda}{d} \tag{12.8}$$

At angle θ_3 in Fig. 12-11*c*, the situation is similar to that of Fig. 12-11*b* in that all waves between A and B will destructively interfere with corresponding waves between B and C and all the waves between C and D interfere destructively with those between D and E. Thus, a dark band is observed for the condition $\sin\theta_3 = 2\lambda/d$. Somewhere in between these two minima a maximum will be observed. We can generalize this result by stating that the diffraction minima occur for angles satisfying the relation

$$\sin\theta = n\frac{\lambda}{d}, \quad \text{where } n = 1, 2, 3, \ldots \tag{12.9}$$

From Eq. 12.8, we can see that if $d = \lambda$, then $\sin\theta_1 = 1$ and $\theta_1 = 90°$, which means that the central maximum spreads over the entire screen. The same will be true if $d < \lambda$. This is what occurs when the screen is illuminated with a point source. We see, therefore, that an opening can be considered a point source when its size is equal to or smaller than the wavelength. Another important conclusion can be drawn from Eq. 12.9. This equation, which gives the position of the *minima* of single slit diffraction, is identical to Eq. 12.6, which gives the angular position of the interference *maxima* for the double slit case. We found then that if the separation between the slits, d, was much greater than the wavelength, the interference pattern would be difficult to observe. The same conclusion holds here if the size of the slit is much greater than λ.

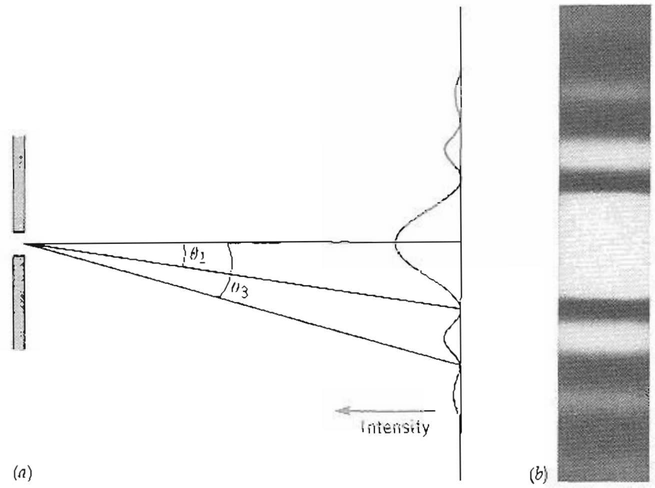


FIGURE 12-12 (*a*) Schematic representation of the interference pattern produced on a screen by light passing through a slit (*b*) Photograph of the interference pattern observed on the screen. (*Source:* Cagnet et al., *Atlas of Optical Phenomena*, New York, Springer-Verlag, 1971.)

The patterns of light on a screen from single and double slits are quite distinctive, as shown in Fig. 12-13.

12.6 RESOLVING POWER

We have seen in the preceding section that light effectively bends around corners. That is, when light shines on a slit the edges of the slit are not simply shadowed on the screen; there are also small bright lines within the shadow at angles away from the normal. This has profound implications for our determination of the location of a particle. We will show here that because of diffraction effects the accuracy of the determination of the precise location of a particle depends on the wavelength used to "look" at it; the smaller the wavelength employed, the greater the accuracy. In Chapter 19 we will show that the smaller the wavelength of the light used, the greater the momentum transferred to the particle being examined and, correspondingly, the

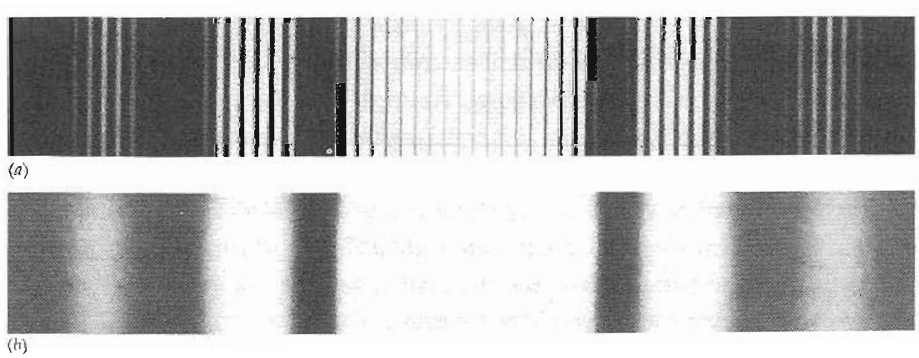


FIGURE 12-13 (*a*) Fringe pattern obtained with light shined through a double slit when the size of the slits is greater than but of the same order of magnitude as the wavelength. The pattern is essentially a double slit interference pattern, similar to that of figure 12-8*b*, "modulated" by the diffraction pattern of Fig. 12-12. (*b*) Single slit diffraction pattern obtain when one of the slits is blocked.

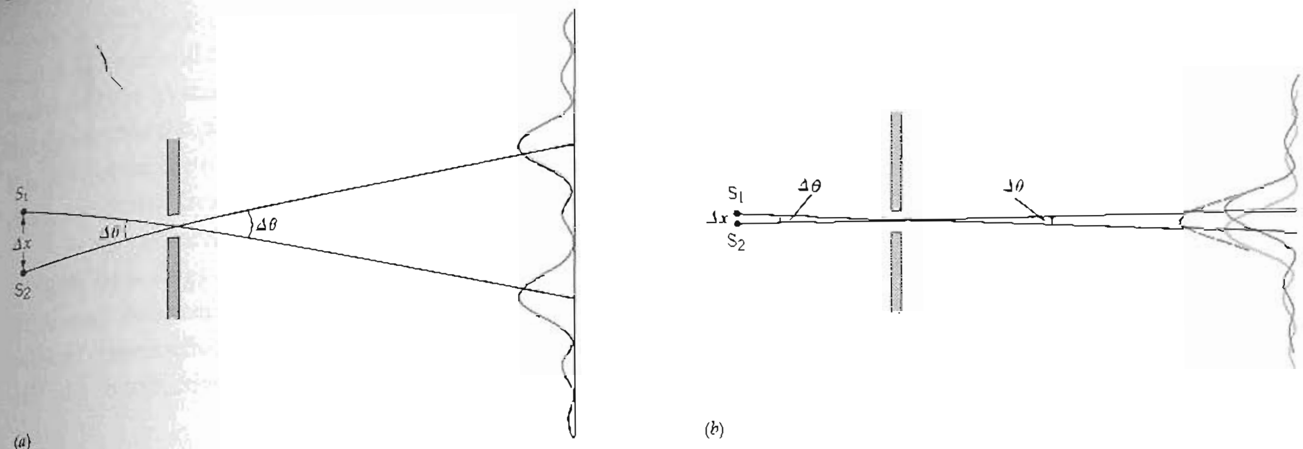


FIGURE 12-14 Two light sources shining on a single slit give rise to two individual diffraction patterns. (*a*) The sources are sufficiently apart and the two central maxima are well resolved. (*b*) The sources are too close together and as a result the two sources cannot be resolved. The two central maxima overlap to the extent that a single intensity maximum (dashed line) appears on the screen.

greater the disturbance of its position in space. Thus, because of these conflicting effects there is a limit to which we may simultaneously know the location and momentum of a particle. This limit is known as the *Uncertainty Principle*. For now, we will show that the determination of location depends on the wavelength; this is called the *resolving power*.

Suppose we have two sources of light, S_1 and S_2, shining on a single slit. If they are sufficiently far apart we will see on a screen two single slit patterns, each of the type of Fig. 12-12. This is illustrated in Fig. 12-14*a*. The light intensity at any point on the screen will be the sum of the contributions from each source. If we know the distance of the sources from the slit and the distance of the screen from the slit, we can calculate the angle $\Delta\theta$ and the distance Δx between the two sources from the separation of the two central maxima. Suppose that the sources are closer together than in Fig. 12-14*a* and we have the arrangement of Fig. 12-14*b*. We see that the bright central maxima of the diffraction pattern from the two sources overlap so that, because it is the sum that is seen, it is difficult to estimate their distance apart and therefore difficult to know Δx. A rather arbitrary, although practical, criterion used to decide when the two sources S_1 and S_2 are just resolvable (i.e., considered as separate sources) is the coincidence of the central maximum of one of them with the first minimum of the other (see Fig. 12-15). This is known as the *Rayleigh criterion* or the *limit of resolution*. We saw in Eq. 12.8 that the first minimum occurs when $\sin\theta_1 = \lambda/d$. Because this occurs for small angles, $\sin\theta_1 \approx \theta_1$ (in radians) and

$$\Delta\theta = \theta_1 \text{(in radians)} = \text{limiting angle of resolution} \approx \frac{\lambda}{d} \qquad (12.10)$$

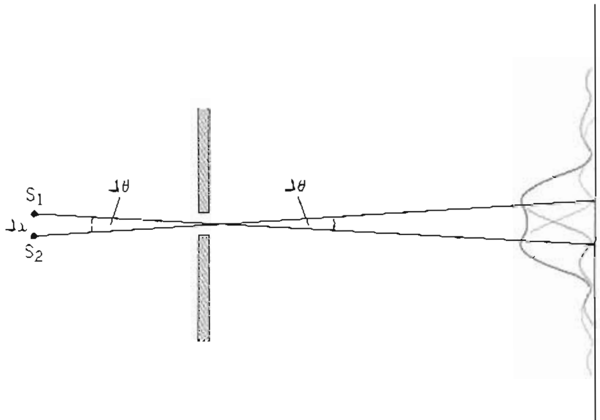


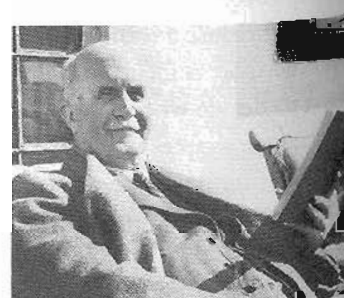
FIGURE 12-15 Rayleigh criterion. The separation between the sources is such that the central maximum produced by one source coincides with the first diffraction minimum from the other. The two sources are barely resolvable.

Thus, for high resolution—that is, small Δx (see Fig. 12-15)—one should have a small λ and a large slit width d.

12.7 X-RAY DIFFRACTION BY CRYSTALS: BRAGG SCATTERING

Waves are scattered or reflected by objects their own size or larger. A wave at the seashore will not be affected by a stick in the water but will be by a large rock or jetty. Atoms have sizes of the order of a few angstroms, 1 to 3×10^{-10} m. Visible light has wavelengths of a few thousand angstroms. Therefore, visible light will not be affected by a single atom. Light is only a small part of the wavelength range of what is called the *electromagnetic spectrum* (Chapter 16). This spectrum ranges between γ-rays and radio waves. The smallest wavelengths that we can conveniently produce are those of X rays, whose wavelengths are about the sizes of atoms. These are produced by bombarding a metal target with high-energy electrons. Their origin will be discussed in Chapter 17.

In 1912, Max von Laue noted that in a crystalline solid the interatomic separation is of the same order of magnitude as the wavelength of X rays. He then showed that the regular, periodic arrangement of atoms in a crystalline solid could be used as a three-dimensional *diffraction grating* (a diffraction grating was defined at the end of Section 12.4). One year later, Sir William Bragg presented a similar but simpler analysis of the problem. We will now present an outline of Bragg's analysis.



Sir William Henry Bragg (1862–1942).

Figure 12-16 is a planar representation of a three-dimensional cubic crystal, that is, a solid in which the atoms are located at the corners of unit cubes. The interatomic separation is d. When X rays strike the crystal at an angle θ with respect to a plane of these atoms, the atoms will scatter them in all directions. We will first limit ourselves to the X rays scattered specularly—that is, X rays for which the angle between the scattered beam and the plane of the crystal is the same as the one between the incident beam and the crystal plane, as shown at points A and B in Fig. 12-16. It is clear from the figure that the path difference between the X rays scattered by atom A and atom B is $2l$ because the ray scattered from atom B must travel that extra distance to rejoin the ray scattered from atom A.

From geometric considerations, $l = d\sin\theta$, and the path difference is $2l = 2d\sin\theta$. By analogy to other interference experiments discussed in previous sections, if this path difference is equal to an integral number of wavelengths, the two beams will add constructively; that is, the radiation reflected by two adjacent layers of atoms will add constructively if

$$2d\sin\theta = n\lambda \quad \text{where } n = 1, 2, 3, \ldots \tag{12.11}$$

Equation 12.11 is known as the *Bragg condition*. Obviously, if the waves reflected by the first layer are in phase with those reflected by the second layer, the same will be true for the waves reflected by the second and third, and so on. Thus the condition of Eq. 12.11 guarantees that the radiation reflected by all the atoms in parallel layers of the crystal at the same distance apart will be in phase.

Thus far we have concentrated our attention on the waves that were scattered specularly. Is it possible to have an angle θ', not necessarily equal to the angle of incidence θ, for which the scattered waves recombine constructively? The answer is yes. However, if such an angle exists, it can be shown that a set of atomic planes exists, different from the ones that we have considered, such that with respect to them, the angle of incidence and the angle of scattering are equal. Furthermore, with respect to this new set of atomic planes, the Bragg condition (Eq. 12.11) is satisfied. This situation is illustrated in Fig. 12.17. To have constructive interference for the direction shown in Fig. 12-17, the condition $2d'\sin\phi = n\lambda$ must be satisfied.

We should note that the Bragg equation is very similar to the double slit equation (Eq. 12.6). In the Bragg equation the interatomic spacing plays the same role as the separation between the slits in the double slit equation. Because the interatomic

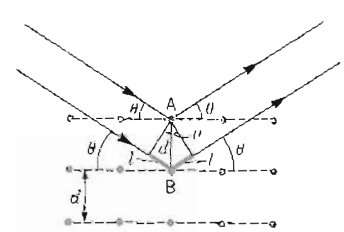


FIGURE 12-16 Planar representation of a three-dimensional cubic crystal of interatomic spacing *d*. The X rays reflected by the dashed atomic planes will recombine constructively if the Bragg condition is satisfied.

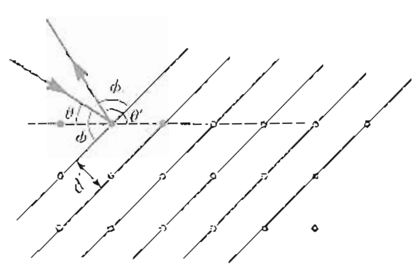


FIGURE 12-17 A different set of atomic planes (indicated by solid lines) in the crystal of Fig. 12-16 can produce constructive reflection if the Bragg condition with respect to those planes is satisfied.

spacing is of the order of 10^{-10} m, we can see how a crystalline solid can be used as a diffraction grating for radiation of small wavelength, such as X rays or, as will be discussed in Chapter 19, matter waves.

EXAMPLE 12-1

The crystal structure of silver bromide (AgBr) is represented in Fig. 12-18. The molecular weight and the density[1] of AgBr are 187.80 g/mole and 6.47 g/cm^3, respectively. (a) Calculate the interatomic separation, d, of the atoms in AgBr. (b) If X rays of wavelength $\lambda = 1.50$ Å are incident on a AgBr crystal, at what angle will the first order ($n = 1$) diffraction maxima occur? Assume that the separation between the atomic planes producing the scattering is the interatomic spacing found in part (a).

Solution

(a) Let us consider a cube of AgBr 1 cm a side. In one row of the cube we have 1 cm/d (cm) atoms. Because we have as many rows of atoms in one plane as there are atoms in a row, we conclude that the number of atoms in one plane of the cube is $1/d \times 1/d$ or $1/d^2$. Finally, we have as many planes of atoms as we have atoms in a row; therefore

$$\text{Number of atoms in 1 cm}^3 = \frac{1\ \text{cm}^3}{d^3}$$

where d is expressed in cm.

The actual number of atoms can be found as follows:

$$N \text{ of atoms/cm}^3 = N \text{ of moles/cm}^3 \times N \text{ of atoms/mole}$$

$$= \frac{6.47\ \text{g/cm}^3}{187.80\ \text{g/mole}} \times 2 \times N_A$$

where N_A is Avogadro's number, 6.02×10^{23} molecules/mole and the factor of 2 is included because there are two atoms (one Ag and one Br) per molecule.

We can now write the following relation

$$\frac{1}{d^3} = \frac{6.47\ \text{g/cm}^3}{187.80\ \text{g/mole}} \times 2 \times 6.02 \times 10^{23}\ \text{molecules/mole}$$

$$= 4.15 \times 10^{22}\ \text{cm}^{-3}$$

therefore

$$d = 2.89 \times 10^{-8}\ \text{cm} = 2.89\ \text{Å}$$

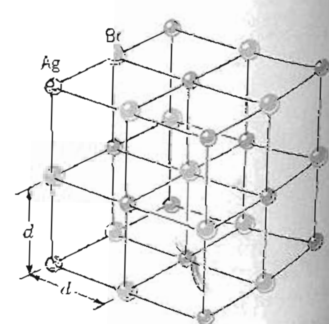


FIGURE 12-18 Example 12-1.

1. Density is defined as mass per unit volume (see problem 13.1).

(b) We can use the Bragg condition, Eq. 12.11, to find the angle at which the first-order ($n = 1$) diffraction maximum is observed.

$$2d \sin\theta = \lambda$$

$$\sin\theta = \frac{\lambda}{2d} = \frac{1.50\text{Å}}{2 \times 2.89\text{Å}} = 0.26$$

$$\theta = \sin^{-1} 0.26 = 15^\circ$$

12.8 STANDING WAVES

Another interesting phenomenon resulting from the superposition principle is the formation of *standing waves*.

In Chapter 11, when we discussed traveling waves in a string, we implicitly assumed that once the waves were set up at one end, they continue traveling toward the right forever. This is a correct assumption if the string is infinitely long. Consider now that the string is of finite length and the other end is clamped to a rigid support. When the wave disturbances reach the fixed end, they will propagate in the opposite direction. The reflected waves will add to the incident waves according to the superposition principle and, under certain conditions, a standing wave pattern will be formed.

If we assume that the incident waves y_1 travel toward the right in the positive x direction, from Eq. 11.5 we can represent them as

$$y_1 = A\sin(kx - \omega t) \qquad (11.5)$$

The reflected waves y_2 will be traveling in the negative x direction and from Eq. 11.7 are given by

$$y_2 = A\sin(kx + \omega t) \qquad (11.7)$$

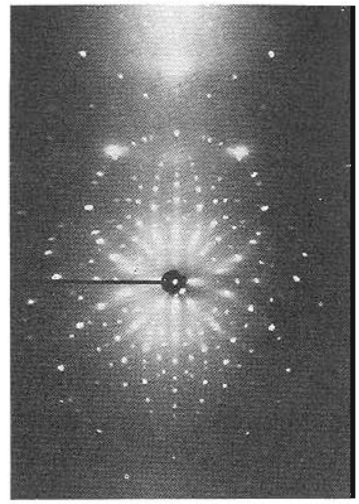
The resulting wave pattern will be

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx + \omega t)]$$

Using the trigonometric relation of Eq. 12.1, we obtain

$$y = 2A \sin kx \cos \omega t \qquad (12.12)$$

Equation 12.12 is the equation of a standing wave. We note that, as in the case of a traveling wave, the particles in the string execute simple harmonic motion with



Diffraction pattern produced when X rays are incident on a NaCl crystal. Each dot is produced by a set of atomic planes, satisfying the Bragg condition.

When this flute player blows on the mouthpiece standing waves are set up inside the flute.

the frequency of the wave $\nu = \frac{\omega}{2\pi}$. Unlike the case of the traveling wave, however, the amplitude of oscillation is not the same for all the points (all values of x in Eq. 12.12) in the string. In particular, there are certain points for which the amplitude of oscillation (the coefficient of $\cos \omega t$) will be zero. These points, called the *nodes*, are those for which $\sin kx = 0$. We can locate these nodes and at the same time find the conditions for standing wave formation by requiring that the value of the wave, y, be zero at the clamped end of the string. If the length of the string is l, then $y(x = l) = 0$. Substituting this in Eq. 12.12, yields

$$0 = 2A \sin kl \cos \omega t \qquad (12.13)$$

Because Eq. 12.13 must be satisfied for all times t, we conclude that

$$\sin kl = 0$$

or

$$kl = \pi, 2\pi, 3\pi, \cdots n\pi \qquad (12.14)$$

where n is an integer. Note that $kl = -\pi, -2\pi, -3\pi, \ldots$ will also yield a zero value for $\sin kl$. However, these negative values correspond to negative values of k and hence of the wavelength λ and therefore are not physically acceptable. Substituting Eq. 11.12 for k in Eq. 12.14, we obtain

$$\frac{2\pi l}{\lambda} = n\pi$$

or

$$\lambda = \frac{2}{n} l \qquad (12.15)$$

This result tells us that the wavelength of the standing wave cannot be arbitrary as was the case with the traveling wave. It can only have the values $2l$, $2/2l$, $2/3l$, $2/4l$, ... A schematic of the first few standing wave patterns is shown in Fig. 12-19.

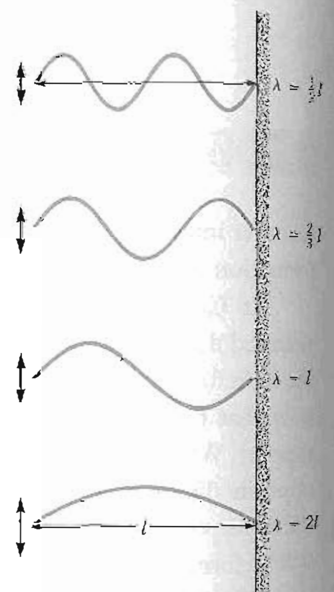


FIGURE 12-19 Configuration of the first four standing waves in a string of length l.

PROBLEMS

12.1 Two sources emit waves of the same frequency, wavelength, and amplitude. What is the amplitude of the resulting wave at a point P at a distance x_1 from source S_1 and a distance x_2 from source S_2 if $x_1 - x_2$ is (a) one wavelength? (b) one-half wavelength?

12.2 Two slits in an opaque screen are separated by a distance $d = 10^{-5}$ m. Light of frequency $\nu = 5 \times 10^{14}$ Hz is shone through the slits. Find the angular position of the first three interference maxima.

12.3 In the double slit of problem 12-2, what is the angular separation between the first interference maxima for two waves of wavelength $\lambda_1 = 6000$ Å and $\lambda_2 = 4000$ Å?

12.4 Two slits separated by a distance $d = 4 \times 10^{-5}$ m, are 1.5 m away from a screen (see Fig. 12-20). What is the separation $y_2 - y_1$ between the first and the second interference maxima if light of wavelength $\lambda = 5000$ Å is sent through the slits?

Answer: 1.88 cm.

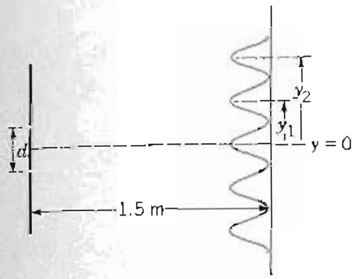


FIGURE 12-20 Problem 12.4.

12.5 Light from a sodium lamp contains waves with wavelength $\lambda_1 = 5880$ Å and waves with $\lambda_2 = 5890$ Å. Find the angular separation and the linear separation of the two wavelengths on a screen 50 cm away from a double slit for the first-order maxima. Do the calculations for a slit separation of 70,000 Å and for a slit separation of 7000 Å.

12.6 In a double slit experiment performed with light of wavelength $\lambda = 5400$ Å, the separation between the tenth interference maximum and the central maximum on a screen 150 cm away is 10 cm. What is the spacing between the slits?

12.7 Two speakers separated by a distance of 3 m emit sound waves of frequency $\nu = 550$ Hz. The velocity of sound is 330 m/sec. Find the position of the points along the line S_1O in Fig. 12-21, at which the intensity of the sound will be a maximum.

Answer: 7.20 m, 3.15 m, 1.60 m, 0.68 m, 0 m from S_1.

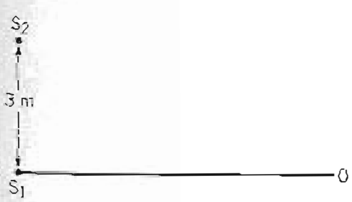


FIGURE 12-21 Problem 12.7.

12.8 A source of waves S and a detector D are located 8 m apart (see Fig. 12-22). A horizontal reflecting surface is placed 3 m above the source and the detector. The direct wave from S to D is found to add constructively with the reflected wave. When the reflecting surface is raised an additional distance of 0.204 m, the direct and the reflected waves add destructively at D. What are the possible values of the wavelength λ?

Answer: 0.500 m, 0.167 m, 0.100 m, 0.071 m, . . .

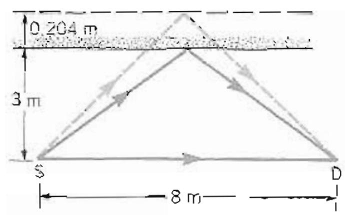


FIGURE 12-22 Problem 12.8.

12.9 Monochromatic (single wavelength) light is directed on a double slit. A light meter is placed to the right of the slits in the position shown in Fig. 12-23. When slit S_2 is closed, the light intensity at the location of the meter is I_1. When slit S_1 is closed the light intensity is I_2. (a) What is the light intensity I_T when both slits are open if $x_1 - x_2 = \lambda$? (b) What is I_T if $x_1 - x_2 = \frac{1}{2}\lambda$? I_1 and I_2 are not necessarily equal. Assume that the size of the slits is smaller than the wavelength so that the slits can be considered to be point sources.

Answer: (a) $I_1 + I_2 + 2(I_1 I_2)^{1/2}$, (b) $I_1 + I_2 - 2(I_1 I_2)^{1/2}$.

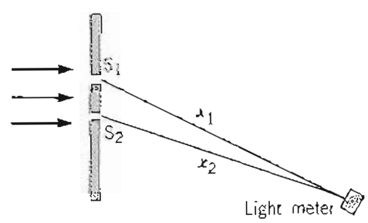


FIGURE 12-23 Problem 12.9.

12.10 Light of wavelength $\lambda = 5000$ Å is incident on a single slit of width 10^{-5} m. What is the angular separation between the central maximum and the eighth-order diffraction minimum?

12.11 Light of wavelength 6000 Å is sent through a single slit. If the angular separation between adjacent diffraction minima is 0.2°, what is the width of the slit? (For small angles, $\sin\theta = \theta$ (in radians)).

Answer: 1.72×10^{-4} m.

12.12 A screen is placed 2 m to the right of a single slit of unknown width. Light of wavelength 5200 Å is incident on the slit from the left. The separation on the screen between the second-order minima on either side of the central maximum is 5.2 cm. What is the width of the slit?

12.13 Monochromatic X rays of wavelength $\lambda = 1.2$ Å are incident on a crystal. The first-order diffraction maximum is observed when the angle θ between the incident beam and the atomic planes is 12°. (a) What is the separation of the atomic planes responsible for the diffraction? (b) What is the highest order Bragg diffraction produced by those planes that can be observed?

Answer: (a) 2.89 Å, (b) 4th.

12.14 Sodium chloride (NaCl) has a crystal structure similar to that of silver bromide (AgBr) shown in Fig. 12-18. The atomic weight of NaCl is 58.44 g/mole and its density is 2.16 g/cm^3. (a) Calculate the spacing between the atoms in a NaCl crystal. (b) If X rays of wavelength 1.5 Å are incident on a NaCl crystal, at what angle θ will the first order diffraction maximum be observed?

Answer: (a) 2.82 Å, (b) 15.4°.

12.15 Potassium chloride (KCl) has the crystal structure of AgBr in Fig. 12-18. The molecular weight and the density of KCl are 74.55 g/mole and 1.98 g/m^3, respectively. The distance between adjacent atomic planes is 3.14 Å. (a) Calculate Avogadro's number from this data. (b) If the first-order diffraction maximum for X rays incident on these atomic planes is observed when the angle θ between the incident direction and the crystal planes is 30°, what is the wavelength of the X rays?

12.16 The wave velocity in a string 1 m long is 6 m/sec. What are the frequencies of the standing waves in the string?

12.17 A standing wave in a string is described by the equation

$$y = (0.7\text{ m})\sin(4\pi x)\cos(20\pi t)$$

where x is in meters and t is in seconds. (a) What is the amplitude and the velocity of propagation of the traveling waves that gave rise to such a standing wave? (b) What is the amplitude of vibration of the particles in the string located at $x = 0.45$ m? (c) What is the transverse velocity of the particles at $x = 0.45$ m at $t = 0.25$ sec? (d) What are the locations of the nodes?

Answer: (a) 0.35 m, 5 m/sec, (b) 0.41 m, (c) 0,
(d) 0 m, 0.25 m, 0.50 m, 0.75 m.

12.18 A standing wave of frequency $\nu = 10$ Hz is set up in a string of mass $m = 0.100$ kg and length $l = 2$ m. The maximum amplitude of vibration is 5 cm. What is the total energy of the standing wave? Assume that $\lambda = l$.

Answer: 0.25 J.

13.1 INTRODUCTION

In this chapter we begin the study of electricity. Although we ultimately wish to understand the flow of electric charges through electrical circuits, we must start with the simple empirical laws of the interaction of charges at rest, called *electrostatics*. Our starting point will be the observed behavior of electric charges at rest and how careful observations by Coulomb led him to postulate laws of the behavior of charges at rest in their interaction with one another. We will also examine the *superposition principle* according to which the behavior of multiple charges on one another is a simple sum of the one-to-one interactions (*pairwise*).

13.2 ATTRACTION AND REPULSION OF CHARGES

Everyone has experienced some of the phenomena of static electricity. When you comb dry, clean hair, it is attracted to the comb. For a short time afterward the comb will attract small particles such as little pieces of paper. These phenomena have been known for thousands of years. The ancient Greeks noticed that if a piece of amber (fossilized tree sap) were rubbed with a piece of cat fur, it would attract pieces of dry leaves. In fact, the Greek word for amber is elektron.

These electrical phenomena fascinated many early investigators, and a useful device called the *electroscope* was invented about 200 years ago to further the studies of these phenomena. An electroscope is simply two very thin gold leaves attached together at the end of a metal rod, usually with a metal knob at the other end. The leaves are enclosed in a protective glass-windowed case (see Fig. 13-1). Gold was used because it is a soft metal that can be beaten very thin. Therefore, the foils have very little mass and not much force is required to push them apart.

If an amber rod (hard rubber or a synthetic polymer will also serve) is rubbed with cat fur and brought close to the metal knob, it is seen that the gold leaves separate into a wide angle as if they are trying to get away from each other. Indeed they are, and their behavior is termed "repelling" one another. If the rod is pulled back from the knob, the leaves collapse again into the downward position. If, however, the rod is touched to the knob before it is removed, the leaves remain apart, in the repelling position. This suggests that something has flowed from the rod to the leaves, and we now know that it is an electric charge. If a glass rod is rubbed with a piece of silk, the same phenomena will occur. However, if the electroscope is first charged with the amber rod and then touched very briefly with the glass rod, it will discharge; that is, the leaves will fall back down. If the glass rod is in contact for a longer period, it is seen that the electroscope will first discharge and then recharge; that is, the leaves will again move apart. This experiment can be done in the reverse. That is, the electroscope

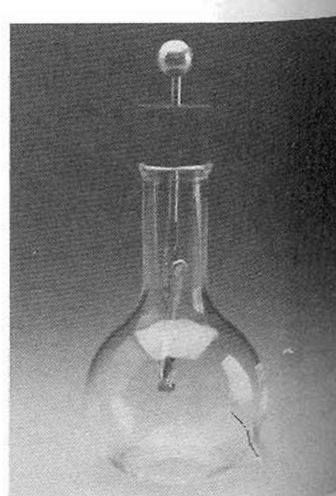
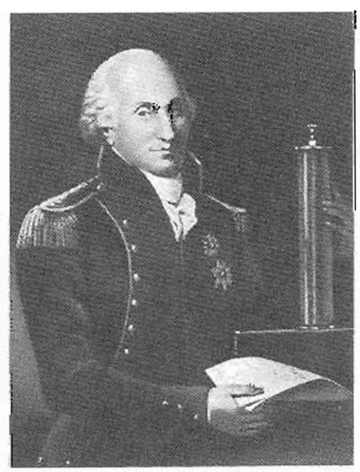


FIGURE 13-1 Electroscope.

charged by the glass rod can be discharged and recharged with the amber rod. These phenomena were explained by the assumption that there are two different types of charges arbitrarily called *positive* and *negative*: one type was produced on rubbing amber with cat fur and the other by rubbing glass with silk. This is still the model today, and we speak of positive and negative charges. In other words, we now say that the electron has a negative charge, but this is a result of history: its charge could just as easily have been called positive without affecting its behavior. It was also noted in these studies that the rod in the electroscope had to be made of metal. No effect would be observed if it were made of wood, rubber, or some other nonmetallic material. This implies that can flow through metal but not through nonmetals, such as the ones just named. Thus metals have been known as electrical *conductors* and nonmetals as *insulators*. In Chapters 24 and 25 we will analyze the difference between these two types of materials and introduce the *semiconductor*, a material with electrical properties that lie between those of a conductor and an insulator and whose characteristics form the basis of the modern computer.

13.3 COULOMB'S LAW

Charles Coulomb (1736–1806) published between 1786 and 1789 the results of a series of experiments that he had performed. Instead of using the electroscope in which the force between the gold leaves was difficult to measure, he used very small lightweight balls on the ends of long threads. The balls were made of the centers of dried reed stems called *pith* and were small so that they could approximate point charges. If two were suspended adjacent to each other and both touched with *either* the amber or glass rod, they would repel each other as did the leaves of the electroscope. He therefore confirmed that *like charges repel*. If he had touched both with the amber rod and achieved repulsion and then touched one with the glass rod they would come together. This experiment could be reversed in that if the repulsion were first achieved by touching both with the glass rod, touching one with the amber rod would cause them to come together. He therefore confirmed that *unlike charges attract*.



Charles Coulomb (1736–1806).

Before continuing with the work of Coulomb, we can use these two findings to interpret the electroscope observations. Accepting that there are two types of charges and that at least one of these can move in a metallic conductor but not in an insulator, we may view the sequence of diagrams in Fig. 13-2. In Fig. 13-2*a*, the negatively charged amber rod is brought near the electroscope and the mobile negative charges in the metal rod, being repelled, go to the ends of the gold leaves, their maximum distance from the amber rod. The gold leaves now both have an excess of negative charge and repel each other. Although at Coulomb's time it was believed that the positive charges were mobile, we now know that this is not generally true and it is the

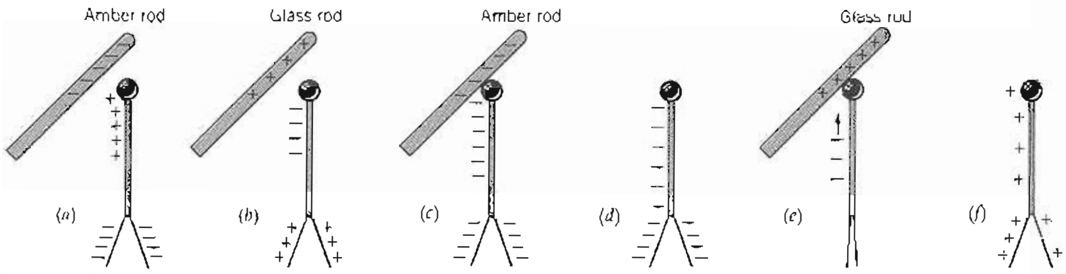


FIGURE 13-2

negatively charged electrons that move in the metal. The same effect occurs in (*b*) with a positively charged glass rod, for it does not matter if the positive charges in the leaves are there because they have been repelled by the glass rod or because the electrons have been attracted to it, thereby leaving a net excess of positive charge on the leaves. In (*c*), the amber rod is touched to the electroscope. Because the upper end of the metal rod, as shown in (*a*), is positively charged, electrons will flow from the amber to the metal rod. When the amber rod is removed in (*d*), the electroscope has a net excess of negative charges throughout and the leaves therefore repel each other. In (*e*), a positively charged glass rod is touched to the negatively charged electroscope of (*d*) and first the excess electrons leave, causing the leaves to collapse because they now have no excess charge, that is, they are neutral. If this positively charged glass rod has a sufficient amount of charge and it is rubbed further against the metal rod of the electroscope, more electrons will leave the electroscope, resulting in an excess of positive charges in the electroscope. Therefore the leaves repel each other as in (*f*).

We now return to Coulomb's experiments concerning the forces between charges. Let us represent a quantity of charge, a scalar, by the letter q and the distance between charges by the letter r. Coulomb, in his experiments, attached fine threads to the pith balls and passed them over glass rods, which effectively served as frictionless pulleys. To the ends of these threads he fastened various weights (see Fig. 13-3*a*). If the charges q_1 and q_2 placed on the pith balls from the amber and glass rods were of opposite sign, the balls would be attracted to each other by a force F, which would be countered by an appropriate weight Mg. The force diagram for q_2 is shown in Fig. 13-3*b*. Note that the tension T_2 in the horizontal thread is equal to the hanging weight Mg. We have here the equilibrium problem of Chapter 4 where the forces to the left are $T_1 \cos\theta$ plus the electrostatic attraction, and these are equal to Mg ($T_2 = Mg$). By measuring the weights Mg required to hold the balls apart a distance r, Coulomb found that the magnitude of the attractive force was proportional to the reciprocal of the square of the distance between the balls; that is,

$$F \propto \frac{1}{r^2}$$

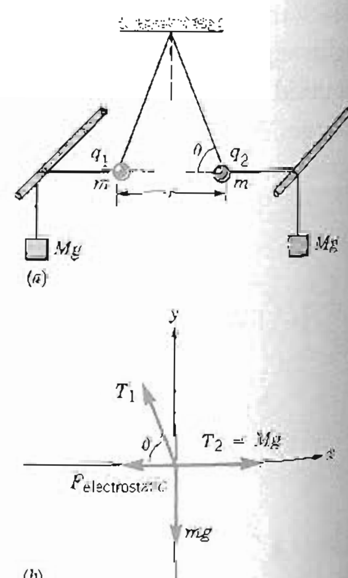


FIGURE 13-3 (*a*) Experimental arrangement for the determination of Coulomb's law. (*b*) Force diagram for q_2.

He then put the same type of charge on each ball and draped the threads over the rods in the opposite direction to measure the force required to hold the balls at various distances r from each other. He found that the force of repulsion also varied inversely as the square of the distance between them. Having established this relation, he was then able to hold them at a fixed distance and vary the charges on them. He could do this accurately by fractionating the charges. He had a set of metal knobs on wooden sticks (insulators). If he touched one of these metal knobs to an amber or glass rod it would acquire a charge of magnitude q. If he then touched this knob to an identical uncharged one they would share the charge equally and each would have a charge $q/2$. Touching either of these to another would reduce the charge to $q/4$, and so on. Touching two knobs in contact with one another would produce a charge of $q/3$. Many combinations of these fractions were used to charge the pith balls at a fixed distance from one another. Coulomb showed that the force of attraction between oppositely charged balls or of repulsion between balls with charges of the same sign was proportional to the product of the magnitude of the two charges, $q_1 q_2$. Coulomb's law is therefore

$$F \propto \frac{q_1 q_2}{r^2} \tag{13.1}$$

where the sign of q_1 and q_2 may be either plus or minus. This proportionality can be made into an equality by introducing a constant dependent on the system of units used. In the SI system this constant, taken as $1/4\pi\epsilon_0$, has the value

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ N-m}^2/\text{C}^2$$

The symbol C stands for Coulomb and is the unit of charge. It is important to note at this time that 1 C is *not* the charge of an electron. The charge of the electron in coulombs is $e = -1.6 \times 10^{-19}$ C.

Equation 13.1 can now be written as

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \tag{13.2}$$

The direction of the force that q_1 exerts on q_2 is along the line joining the two charges, pointing away from q_1 if the force is repulsive (q_1 and q_2 having the same sign) or toward q_1 if the force is attractive (q_1 and q_2 oppositely charged).

EXAMPLE 13-1

Two pith balls of mass 0.1 g each are suspended on 50-cm threads. They are given equal charges and assume a position in which each makes an angle of 20° with the vertical, as in Fig. 13-4*a*. What is the charge on each?

Solution The vector diagram of the forces on the right-hand ball is shown in Fig. 13-4*b*, where F is the coulombic force of repulsion between the two charged pith balls. Because the ball is in equilibrium, we may write

$$\sum F_x = 0$$

$$F - T\cos 70^\circ = 0$$

$$F = 0.34T$$

$$\sum F_y = 0$$

$$T\sin 70^\circ - mg = 0$$

$$T = \frac{mg}{\sin 70^\circ}$$

$$T = \frac{0.1 \times 10^{-3}\,\text{kg} \times 9.8\,\text{m/sec}}{0.94} = 1.04 \times 10^{-3}\,\text{N}$$

Substituting this value of T in the equation for F, we obtain

$$F = 0.34T = 0.34 \times 1.04 \times 10^{-3}\,\text{N} = 3.5 \times 10^{-4}\,\text{N}$$

From Fig. 13-4*a*, the distance r between the two balls is

$$r = 2l\sin 20^\circ$$

$$r = 2 \times 0.5m \times \sin 20^\circ$$

$$r = 0.34m$$

Using Coulomb's law,

$$F = 9 \times 10^9\,\frac{\text{N-m}^2}{\text{C}^2}\,\frac{q^2}{r^2}$$

because

$$q_1 = q_2$$

and substituting for F and r we obtain

$$3.5 \times 10^{-4}\,\text{N} = \frac{9 \times 10^9\,\frac{\text{N-m}}{\text{C}^2}\,q^2}{(0.34\,\text{m})^2}$$

or

$$q = 6.7 \times 10^{-8}\,\text{C}$$

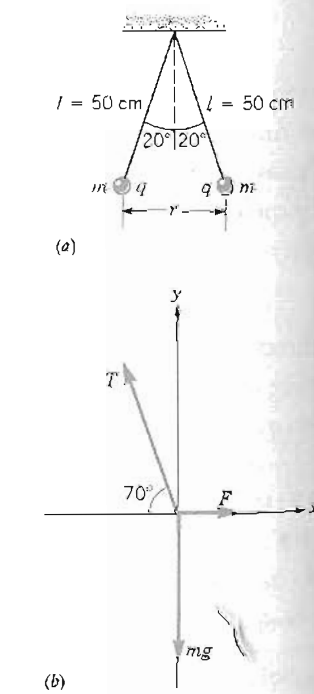


FIGURE 13-4 Example 13-1.

13.4 CHARGE OF AN ELECTRON

In a series of experiments, Robert Millikan (1868–1953) in the years 1909 through 1913 measured the charge on an electron (see Fig. 13-5). With a spray he introduced fine oil drops between two parallel metal plates and observed the motion of a single drop through a telescope. By its rate of fall through the air he was able to use a formula for the terminal velocity (constant rate of fall through a medium) to estimate its weight. He found that he could arrest its downward motion, that is, hold it stationary by placing a positive charge on the upper plate. (We will see later that he could control the positive charge on the plate by means of the voltage.) Therefore, the balance of forces of mg down and the upward attraction could be used to determine the charge on the drop. First, he found that the drops usually acquired a negative charge. This showed that it is the negative charge that is apparently the more mobile one. His second finding, over hundreds of experiments, was that the smallest charge that was ever acquired by the drop had a magnitude of 1.6×10^{-19} C and that larger charges were always integral multiples of this quantity. He therefore assigned this value to the charge of the electron; it is the smallest negative charge that can be found. Because atoms are neutral and contain equal numbers of electrons and protons, the charge of the heavy and essentially immobile proton also has this magnitude, but it is positive. We will see that the protons are in the nucleus and are relatively massive. It is therefore understandable why it is the negative charge that is the mobile species rather than the positive one. This was not known prior to Millikan's experiment. The great theories of electrical behavior were developed in the nineteenth century when it was assumed that the positive charge was the mobile species. We will see that all electrical definitions are based on the behavior of a unit positive charge. This sometimes leads to the confusion of students; but, as mentioned earlier, the assignment of the words positive and negative are completely arbitrary, and therefore the theories remain valid.

Robert Millikan (1868–1953).

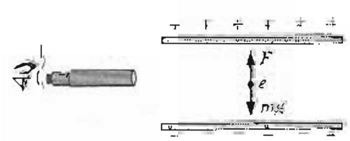


FIGURE 13-5 Experimental arrangement for the determination of the charge of the electron.

13.5 SUPERPOSITION PRINCIPLE

Coulomb's law, defined in Section 13.3, relates to the force between two charges. It is an empirical law derived from experimental measurement. How does one treat a situation in which three or more charges are involved? To answer that requires further experimentation. It is found that the force between any two charges in a group of charges is independent of the presence of the other charges. What this means is that if one selects a given charge in a group and asks for the total force on it, this force would be the resultant of the individual vector forces on it from each of the charges. This is called the *superposition principle* of charges. It makes the calculations straightforward because we can treat them as we did in summing vector forces in Chapter 2. Let us examine the case of three charges in Example 13-2.

EXAMPLE 13-2

Three charges are arranged in a triangle as shown in Fig. 13-6*a*. What is the direction and the magnitude of the resultant force on the 1×10^{-8} C charge?

Solution The force resulting from the 4×10^{-8} C charge by Coulomb's law is

$$F_1 = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \frac{1 \times 10^{-8}\text{ C} \times 4 \times 10^{-8}\text{ C}}{10^{-2}\text{ m}^2} = 3.6 \times 10^{-4}\text{ N}$$

at 30° above the positive x axis (see Fig. 13-6*b*).

The force resulting from the 2×10^{-8} C charge is

$$F_2 = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \frac{1 \times 10^{-8}\text{ C} \times 2 \times 10^{-8}\text{ C}}{10^{-2}\text{ m}^2} = 1.8 \times 10^{-4}\text{ N}$$

at 30° below the positive x axis (Fig. 13-6*b*).

We now use the vector diagram of these two forces, Fig. 13-6*b*, and find the resultant by the component method of Chapter 2. We have

$$\mathbf{F}_1 = 3.6 \times 10^{-4}\text{ N} \cos 30°\mathbf{i} + 3.6 \times 10^{-4}\text{ N} \sin 30°\mathbf{j}$$

$$\mathbf{F}_2 = 1.8 \times 10^{-4}\text{ N} \cos 30°\mathbf{i} - 1.8 \times 10^{-4}\text{ N} \sin 30°\mathbf{j}$$

$$\mathbf{R} = 4.7 \times 10^{-4}\text{ N}\mathbf{i} + 0.9 \times 10^{-4}\text{ N}\mathbf{j}$$

$$|R| = \sqrt{(4.7 \times 10^{-4}\text{ N})^2 + (0.9 \times 10^{-4}\text{ N})^2} = 4.8 \times 10^{-4}\text{ N}$$

$$\theta = \arctan \frac{0.9}{4.7} = 10.8° \text{ above the positive } x \text{ axis}$$

The direction of the resultant force vector is indicated by the arrow labeled R in Fig. 13-6*a*.

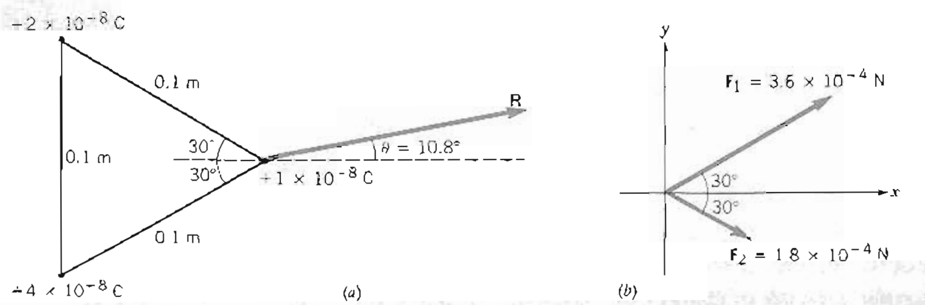


FIGURE 13-6 Example 13-2

PROBLEMS

13.1 Two particles of charge $q_1 = +2 \times 10^{-9}$ C and $q_2 = +3 \times 10^{-9}$ C are placed 0.04 m from each other. What is the force of repulsion that each experiences?

13.2 A particle of charge $q_3 = -2 \times 10^{-9}$ C is placed midway between the two charged particles of problem 13.1. What is the net force on it and in what direction?

13.3 At what position between particles 1 and 2 of problem 13.1 will particle 3 of problem 13.2 experience no net force?

Answer: 0.018 m from q_1, between q_1 and q_2.

13.4 Three charges lie on the x axis as in Fig. 13-7. Find the resultant force on the middle charge, q_2.

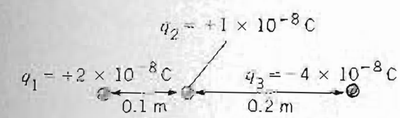


FIGURE 13-7 Problem 13.4.

13.5 An iron atom of mass 9.32×10^{-26} kg has 26 electrons. The density of iron is 7.86 g/cm^3. If two identical iron balls each of volume 1 cm^3 were stripped of all their electrons and placed 1 m apart, (a) What would be the electrostatic force between them? (b) What would be the gravitational force between them? (c) Compare these forces.

13.6 Two particles of mass 5 kg each are given an equal amount of charge. (a) What must the charge be on each particle so that the gravitational attraction exactly balances the electrostatic repulsion? (b) How many electronic charges does that charge correspond to?

13.7 Two threads of length 0.7 m support balls of mass 0.2 gm as in Example 13.1. Equal charges of the same sign are put on the balls and they repel, each making an angle of 30° with the vertical. What is the charge on each ball?

Answer: 2.48×10^{-7} C.

13.8 A charge q is to be shared by two particles. What must be the charge on each particle so that the force between them, for a fixed separation, is a maximum?

Answer: $\frac{q}{2}$.

13.9 An *electric dipole* consists of two charges of equal magnitude q but of opposite sign separated by some distance d. The electric dipole moment is defined as $\mu_e = qd$. Consider an electric dipole lying on the y axis as in Fig. 13-8. What is the force exerted by the dipole on a charge q' located on the x axis at a distance x from the origin?

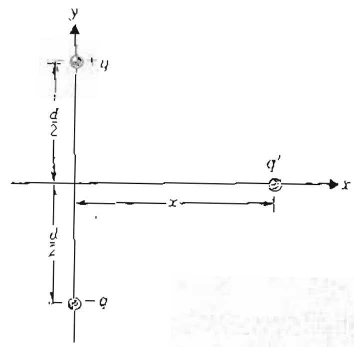


FIGURE 13-8 Problem 13.9.

13.10 Four charges are located at the corners of a square as shown in Fig. 13-9. (a) What is the resultant force on q_2? (b) What should q_1 and q_4 be so that the resultant force on q_2 is zero?

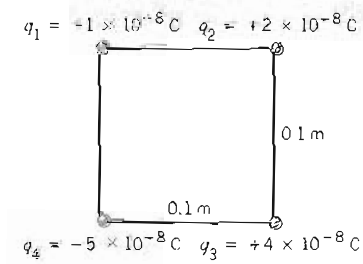


FIGURE 13-9 Problem 13.10.

13.11 Two charges, $q_1 = 3 \times 10^{-6}$ C and $q_2 = -3 \times 10^{-6}$ C, are connected by an insulating rod 10 m long. The rod is pivoted about its midpoint. The rod is kept horizontal 10 cm above the floor. Two identical charges, $q = 5 \times 10^{-6}$ C, are fixed directly below q_1 and q_2, as shown in Fig. 13-10. Where should a 3-kg weight be placed on the rod to keep the rod horizontal?

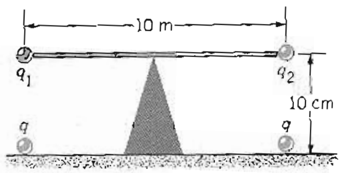


FIGURE 13-10 Problem 13.11.

13.12 A fixed conducting ball has a charge $q_1 = 3\times10^{-6}$ C. An identical ball with charge q_2 is held at a distance x away from q_1. The two balls attract each other with a force of 13.5 N. The balls are then connected by a conducting wire. After the wire is removed, the balls repel each other with a force of 0.9 N. (a) What was the charge q_2 of the second ball? (b) What is the separation x between the balls?

Answer: (a) -5×10^{-6} C, (b) 0.10 m.

13.13 An α particle ($q = 3.2\times10^{-19}$ C) is projected from far away directly toward a gold nucleus ($q' = 79\times1.6\times10^{-19}$ C). The mass of the α particle is 6.7×10^{-27} kg and its initial velocity is 4×10^6 m/sec. Use the work-energy theorem of Section 5.4 (Eq. 5.9) to calculate the closest distance of approach of the α particle to the nucleus. Assume that the gold nucleus remains stationary. (*Hint*: In this case the force on the α particle is not constant. Therefore, the integral form for work, Eq. 5.7′, must be used to evaluate the work done on the α particle.)

Answer: 6.8×10^{-13} m.

13.14 Two positive charges q are held fixed and are separated by a distance $2a$. A third positive charge q' of mass m is initially placed halfway between them (Fig. 13-11). q' is then displaced a small distance x ($x \ll a$) and released. (a) Show that the force on q' is approximately proportional to x and in the opposite direction of the displacement x. (b) The force on q' is therefore similar to the force exerted by a spring on a block connected to it (Chapter 10). What is the period of oscillation of q'?

Answer: $2\pi\left[\dfrac{\pi\epsilon_0 a^3 m}{qq'}\right]^{1/2}$.

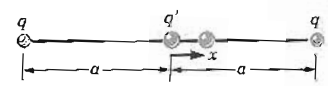


FIGURE 13-11 Problem 13.14.

14.1 INTRODUCTION

We have seen in the preceding chapter how the presence of an electric charge has an effect on another electric charge. This raises the question: What if there is only one electric charge present? The idea of an *electric field* is introduced to describe the effect in all space around a charge so that if another charge is present we can predict the effect on it. If we have multiple charges, such as in Example 13-2, we see that the force of each on a third charge is a vector, and the net effect on the third charge is the resultant of the forces. This resultant will differ with both the position and the charge of the third one. The concept of separating the calculation into the formation of an electric field and the response to the electric field by a given charge placed in it greatly simplifies the calculations.

14.2 THE ELECTRIC FIELD

If, in Example 13-2, we had wished to find the resultant force on a charge of a different magnitude in the same position, we would have to repeat the calculations. However, after a few such repeat calculations we would notice that all one has to do is multiply the first result by the ratio of the magnitude of the new charge to that of the charge used in the first calculation. If the first calculation has been made for a test charge of +1 C, the task will be easier, for then the ratio of the magnitude of the new charge to that of the charge of the first calculation is simply the magnitude of the new charge. In other words, let us simply define a new quantity, called the *electric field*, with symbol $\mathcal{E}$, at a point in space as the vector resultant force experienced by a *positive* test charge of magnitude 1 C placed at that point. If an arbitrary test charge q' is placed at that point, the charge will experience a force

$$\mathbf{F} = q'\mathcal{E} \tag{14.1}$$

Thus, the electric field at a point in space can be calculated by measuring the force experienced by a test charge q' and dividing it by the magnitude of the test charge; that is,

$$\mathcal{E} = \frac{\mathbf{F}}{q'} \tag{14.2}$$

If we consider two charges, q and q', separated by a distance r, from Coulomb's law (Eq. 13.2) the magnitude of the force between them is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2} \tag{13.2}$$

Let us arbitrarily consider q' as the test charge and q as the charge creating the electric field at the point P where q' is located. On substitution of Eq. 13.2 in Eq. 14.2, the magnitude of the electric field produced by q at P is given by

$$\mathcal{E} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{q'r^2}$$

or

$$\mathcal{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \tag{14.3}$$

Equation 14.3 is the general expression for the electric field resulting from a charge q at a point located a distance r away from the charge.

The direction of $\mathcal{E}$ can be deduced from its original definition as the force experienced by a unit positive charge. If the charge q producing the field is positive, then a positive test charge of 1 C, when placed at P, will be repelled. We conclude that the electric field produced by a positive charge q at a point P is along the line joining the charge q and the point P and directed away from q. It may be said that the electric field is directed radially away from a point charge q (see Fig. 14-1*a*). On the other hand, a negative point charge q will attract the 1-C positive test charge and consequently the electric field that it produces is directed radially toward it (see Fig. 14-1*b*).

We have indicated (Section 13.5) that Coulomb's law obeys the superposition principle. From the definition it follows that the electric field does too. The field produced by a group of charges is simply the vector sum of the fields produced by the individual charges.

An important point to mention here is that not only is there no electric field when there are no charges, but there is no electric field at a point when the force from an assembly of charges on a test charge is zero at that point. Also, if an array of equal numbers of positive and negative charges are located in a small region, then at some distant point (distant relative to the distance between the charges) there is no measurable electric field. This is why atoms when they are far away from each other, such as in a dilute gas, experience no measurable electric fields. However, the electrons that make up the atom experience the electric fields of the nucleus and of the other electrons. Furthermore, if a charged particle is shot at an atom, as in a nuclear experiment, it will get close enough to experience the internal electric fields of the atom.

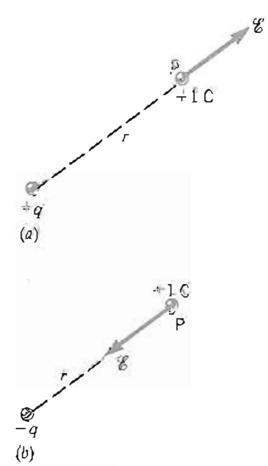


FIGURE 14-1 Direction of the electric field set up at point P: (*a*) by a positive charge q, and (*b*) by a negative charge $-q$.

EXAMPLE 14-1

A charge $q_1 = 3 \times 10^{-6}$ C is located at the origin of the x axis. A second charge $q_2 = -5 \times 10^{-6}$ C is also on the x axis 4 m from the origin in the positive x direction. (a) Calculate the electric field at the midpoint P of the line joining the two charges. (b) At what point P' on that line is the resultant field zero?

Solution

- (a) Because q_1 is positive, its electric field $\mathcal{E}_1$ at P is away from it, that is,

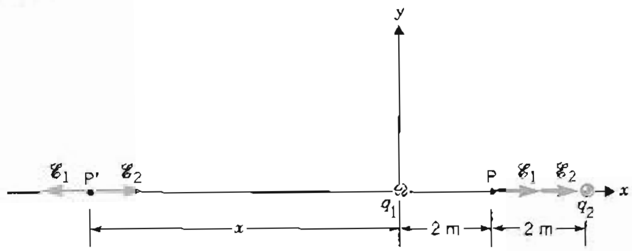


FIGURE 14-2 Example 14-1.

in the positive x direction. The electric field $\mathcal{E}_2$ produced at point P by q_2 is toward q_2, that is, in the same direction as $\mathcal{E}_1$ (see Fig. 14-2). From Eq. 14.3, the magnitudes of $\mathcal{E}_1$ and $\mathcal{E}_2$ are

$$|\mathcal{E}_\infty| = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2}\frac{3 \times 10^{-6}\text{C}}{(2\,\text{m})^2} = 6.75 \times 10^3\,\text{N/C}$$

$$|\mathcal{E}_\in| = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2}\frac{5 \times 10^{-6}\text{C}}{(2\,\text{m})^2} = 11.25 \times 10^3\,\text{N/C}$$

Because it is seen in Fig. 14-2 that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are directed along the positive x direction, the resultant electric field $\mathcal{E}$ at P will be

$$\mathcal{E} = \mathcal{E}_\infty + \mathcal{E}_\in = \not\nabla \times \infty/^{\ni}\,\text{N/C} + \infty\infty.\in\nabla \times \infty/^{\ni}\,\text{N/C}$$

$$= 18 \times 10^3\,\text{N/C}$$

(b) From part (a), it is clear that the resultant $\mathcal{E}$ cannot be zero at any point between q_1 and q_2 because both $\mathcal{E}_1$ and $\mathcal{E}_2$ are in the same direction. Similarly $\mathcal{E}$ cannot be zero to the right of q_2 because the magnitude of q_2 is greater than q_1 and the distance r in Eq. 14.3 is smaller for q_2 than q_1. $\mathcal{E}$ can only be zero to the left of q_1 at some point P′ to be found.

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = 0$$

$$\mathcal{E}_1 = -\mathcal{E}_2$$

and

$$|\mathcal{E}_1| = |\mathcal{E}_2|$$

or

$$\frac{1}{4\pi\epsilon_0}\frac{q_1}{x^2} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{(x+4)^2}$$

$$3(x+4)^2 = 5x^2$$

$$2x^2 - 24x - 48 = 0$$

$$x = 13.75\ \text{m}, \quad x = -1.75\ \text{m}$$

The second root of the quadratic equation, $x = -1.75$ m, represents a point between the charges at which $\mathcal{E}_1 = \mathcal{E}_2$. However, as indicated earlier, between the charges, $\mathcal{E}_1$ and $\mathcal{E}_2$ have the same direction and consequently the resultant field is not zero.

14.3 ELECTRICAL POTENTIAL ENERGY

We now wish to develop an expression for the amount of work required to move a charge in an electric field. First we note that the magnitude of the electric field at a point P resulting from a point charge is independent of the angular position of the point P, because only distance enters into Eq. 14.3. In the preceding section we also showed that the direction of the electric field is radially away from the charge producing the field if the charge is positive or radially toward it if the charge is negative. Thus, the direction of the electric field from a positive point charge may be represented by simply drawing arrows out from it in all directions. Figure 14-3 shows such a schematic in two dimensions only. If we move a positive test charge q' from point A to point B, we have a situation similar to that of Chapter 5 where we showed that the work against a gravitational force is independent of the path. We must recognize that in this case the force is not constant. In Fig. 14-3 work must be done whenever the radial distance between the moving charge, q', and that which creates the electric field, q, is changed. However, when q' moves tangentially, no work is done because the direction of motion is perpendicular to the electric field and therefore to the force acting on q'. Recall that by definition work involves the dot product of the force vector **F** and displacement vector $\Delta \mathbf{s}$, that is, $W = \mathbf{F} \cdot \Delta \mathbf{s}$ (Eq. 5.3). In Fig. 14-3 the same amount of work is done in moving a charge from point A to point B either by path 1 (solid line) or by path 2 (dashed line) or by any other path.

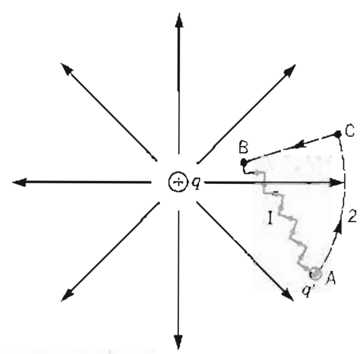


FIGURE 14-3 Two possible paths for bringing a charge q' from point A to point B.

Because the force on the test charge q' is not constant but changes with distance, we use Eq. 5.7′ to evaluate the work done in moving it from point A to point B

$$W_{A \to B} = \int_A^B \mathbf{F} \cdot d\mathbf{s} \qquad (5.7')$$

Just as in the case of gravity, considered in Chapter 5, the force **F** needed to move q' at constant velocity must be equal and opposite to the force exerted by the electric field of q, that is, because the force of the electric field on q' is $q'\mathcal{E}$, a force $\mathbf{F} = -q'\mathcal{E}$ is needed to move q' with constant velocity, where $\mathcal{E}$ is the electric field produced by q. Substituting this for **F** in Eq. 5.7′ we have

$$W_{A \to B} = -q' \int_A^B \mathcal{E} \cdot d\mathbf{s} \qquad (14.4)$$

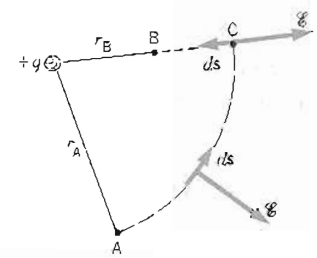


FIGURE 14-4

Because $W_{A \to B}$ is independent of the path followed, we will evaluate it by moving tangentially from point A to C and then radially from point C to point B (see Fig. 14-4). During the first leg of this trip (A to C), the work done is zero because $\mathcal{E}$ is perpendicular to the displacement ds. However, the incremental work done along the path from C to B is $\mathcal{E} \cdot ds = \mathcal{E}\, ds \cos 180° = -\mathcal{E}\, ds$, and Eq. 14.4 becomes

$$W_{A \to B} = q' \int_{C}^{B} \mathcal{E}\, ds \qquad (14.5)$$

As we move a distance ds toward B from point C, the radius r decreases, which introduces a negative sign, that is, $ds = -dr$. Using this in Eq. 14.5, we obtain

$$W_{A \to B} = -q' \int_{C}^{B} \mathcal{E}\, dr \qquad (14.6)$$

The electric field for a point charge q is given by Eq. 14.3. Substitution of Eq. 14.3 for $\mathcal{E}$ in Eq. 14.6 yields

$$W_{A \to B} = -\frac{qq'}{4\pi\epsilon_0} \int_{r_A}^{r_B} \frac{dr}{r^2}$$

Note that we have put r_A instead of r_C as the lower limit of the integral because $r_A = r_C$ and we are evaluating the work done in moving q' from A to B. Integrating obtains

$$W_{A \to B} = \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \qquad (14.7)$$

By definition (see Section 5.3), the work done in moving an object between two points in a force field is equal to the difference in the potential energy E_p between the two points; that is,

$$E_p(B) - E_p(A) = \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \qquad (14.8)$$

Equation 14.8 gives the difference in the potential energy of the two charges when q' is located at two different points. It does not give the potential energy of the charges when q' is at B or at A. For this, as indicated in Section 5.3, we must specify a reference point, that is, a point at which the potential energy is arbitrarily chosen to be zero. In electrostatics, this point is often chosen to be $r = \infty$, that is, when the two charges are separated by an infinite distance. With this assumption, the potential energy of our two charge system q and q' when they are separated by a distance r is simply the work done in bringing one of them (for example q') from infinity to r. Setting $r_A = \infty$ and $r_B = r$ in Eq. 14.8, we have

$$E_p(r) = \frac{1}{4\pi\epsilon_0} \frac{qq'}{r} \qquad (14.9)$$

This potential energy is called *electric potential energy* to differentiate it from the gravitational or the elastic potential energies that we encountered earlier. We should note that, because both q and q' are positive, E_p is also positive. To move q' from infinity to r we have to do positive work, we have to overcome the repulsive force between the two charges, that is, the external force must act in the direction of the displacement.

The same is true if both q and q' are negative. Equation 14.9 holds also in this case because the product of two negative charges will yield a positive value for E_p. If the charges are of unlike sign, they will attract each other and, consequently, to move q' at constant velocity, we will have to hold it back. We will then do negative work and therefore, the potential energy will be negative. It is seen that Eq. 14.9 agrees with this, for if q is positive and q' is negative, or vice versa, E_p will be negative.

Let us now have two fixed charges q_1 and q_2 at a distance r_{12} from each other. To achieve this, an amount of work equal to

$$\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}}$$

had to be done; that is, the potential energy of the charges is

$$E_p = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}} \tag{14.10}$$

Consider now a third charge q_3 that is brought from infinity to point P as shown in Fig. 14-5. How much work must be done? Or equivalently, what is the change ΔE_p in the potential energy of the charges? From Eq. 14.4, setting $A = \infty$, $B = P$, and $q' = q_3$, we have

$$\Delta E_p = -q_3 \int_\infty^P \mathcal{E} \cdot ds \tag{14.11}$$

We have already indicated that the electric field obeys the superposition principle. That is, $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the electric fields produced by q_1 and q_2, respectively. Substituting for $\mathcal{E}$ in Eq. 14.11, we obtain

$$\Delta E_p = -q_3 \int_\infty^P (\mathcal{E}_1 + \mathcal{E}_2) \cdot ds$$

or

$$\Delta E_p = -q_3 \int_\infty^P \mathcal{E}_1 \cdot ds - q_3 \int_\infty^P \mathcal{E}_2 \cdot ds \tag{14.12}$$

Equation 14.12 shows that the total work done in bringing q_3 to point P is simply equal to the sum of the work done against the electric field produced by each charge in the absence of the other. Thus, by using Eq. 14.9 for q_1 and q_3 and for q_2 and q_3 independently, we may write

$$\Delta E_p = \frac{1}{4\pi\epsilon_0}\frac{q_1q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2q_3}{r_{23}} \tag{14.13}$$

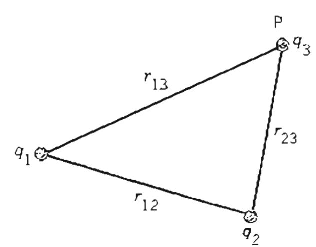


FIGURE 14-5

We should remember that energy is a scalar quantity and, therefore, the sum of the two contributions to the potential energy in Eq. 14.13 is an algebraic sum, not a vector sum as in the case of the electric field. The total energy of the three-charge system shown in Fig. 14-5 is obtained by combining Eq. 14.13 with Eq. 14.10, that is,

$$E_p = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) \tag{14.14}$$

Thus, for a system of charges, the procedure to follow is to calculate the potential energy separately for the pairs and then to add these algebraically.

EXAMPLE 14-2

Three charges—$q_1 = 3 \times 10^{-6}$ C, $q_2 = -5 \times 10^{-6}$ C, and $q_3 = -8 \times 10^{-6}$ C—are positioned on a straight line as shown in Fig. 14-6. Find the potential energy of the charges.

Solution From Eq. 14.14, we may write

$$E_p = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2}\left[\frac{(3 \times 10^{-6}\,\text{C})(-5 \times 10^{-6}\,\text{C})}{4\,\text{m}}\right.$$

$$+ \frac{(3 \times 10^{-6}\,\text{C})(-8 \times 10^{-6}\,\text{C})}{9\,\text{m}}$$

$$\left. + \frac{(-5 \times 10^{-6}\,\text{C})(-8 \times 10^{-6}\,\text{C})}{5\,\text{m}}\right]$$

$$E_p = 1.43 \times 10^{-2}\,\text{J}$$

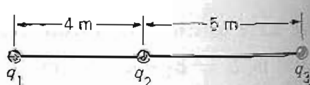


FIGURE 14-6 Example 14-2.

14.4 ELECTRIC POTENTIAL

In the preceding section we saw that when a test charge q' is moved from point A to point B work is done against the electric field produced by q. The amount of work done (Eq. 14.4) depends on the strength of the field and on the magnitude of the test charge q'. We can introduce a quantity, called the *electric potential*, with symbol V, which is independent of the test charge. The electric potential at a point P is defined as the work done in bringing a unit positive charge from infinity to the point. That is, from Eq. 14.4, setting A $= \infty$, B $=$ P, and $q' = +1$ C, we have

$$W_{\infty\to P} = V(P) = -\int_{\infty}^{P} \mathcal{E} \cdot ds \tag{14.15}$$

Alexander Volta (1745–1827).

Because the magnitude of the test charge q' was set equal to unity, the potential at a point depends on the electric field alone and not on the test charge q'. However, if we know the potential at point P, we can immediately conclude, by comparing Eq. 14.4 and Eq. 14.15, that the work done in bringing a charge q' of arbitrary magnitude or sign to P is $W = q'V(\mathrm{P})$. But this work, by definition, is the potential energy of the charge, and we therefore write

$$E_p = q'V \tag{14.16}$$

The SI unit of potential is known as the *volt* in honor of the Italian scientist Alexandro Volta (1745–827). From Eq. 14.16 it is seen that *one volt* can be defined as *one joule per coulomb*.

We can use the results of the preceding section to evaluate the potential resulting from a point charge q at a distance r away from it. If we equate Eqs. 14.9 and 14.16, we conclude that

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{14.17}$$

In the preceding section, we saw that the work done in moving a charge in the resultant field of several charges could be found by summing the work done against the electric field independently produced by each charge. We therefore conclude that the potential resulting from several point charges is simply equal to the *algebraic* sum (remember that work is a scalar quantity) of the potential resulting from each charge. That is,

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} + \cdots \tag{14.18}$$

where $r_1, r_2, \ldots$ are the distances from q_1 and q_2, respectively, to the point where the potential is being evaluated.

In electricity, as in mechanics, one is often interested in the *difference* in potential between two points rather than in the absolute value of the potential at a point. A potential difference between two points is commonly referred to as a *voltage difference* or simply *voltage*. This difference can be found by applying Eq. 14.18 to the two points in question and finding the difference. Alternatively, the potential difference can be calculated directly from the electric field. From Eq. 14.4 the work done in moving q' from A to B is

$$W_{\mathrm{A}\to\mathrm{B}} = -q'\int_{\mathrm{A}}^{\mathrm{B}} \mathcal{E}\cdot ds \tag{14.4}$$

and by definition, this is equal to the difference in potential energy $E_p(\mathrm{B})\,E_p(\mathrm{A})$. Using the relation between potential and potential energy, Eq. 14.16, we write

$$\frac{W_{\mathrm{A}\to\mathrm{B}}}{q'} = \Delta V = V(\mathrm{B}) - V(\mathrm{A}) = -\int_{\mathrm{A}}^{\mathrm{B}} \mathcal{E}\cdot ds$$

We can eliminate the minus sign by inverting the limits of integration, that is,

$$\Delta V = V(\mathrm{B}) - V(\mathrm{A}) = \int_{\mathrm{A}}^{\mathrm{B}} \mathcal{E}\cdot ds \tag{14.19}$$

Several practical conclusions may be drawn from Eq. 14.19. Consider two plates, B which is positively charged and A which is negatively charged (see Fig. 14-7). As we saw in Section 14.2, the electric field is directed away from the positive charges and toward the negative charges. Thus in Fig. 14-7, $\mathcal{E}$ is directed from plate B to plate A. A unit of positive charge placed at B will be accelerated toward A. Noting that objects are accelerated when they move from a point to another of lower potential energy (recall the case of gravity), and remembering that by definition, the potential at a point is the potential energy of a unit of positive charge at that point, we conclude that $V(\mathrm{B}) > V(\mathrm{A})$. That is, the positively charged plate is at a higher potential than the negatively charged one. The result illustrated in this example can be generalized by stating that *the electric field is directed from high potential points to low potential points*, and that *positive charges, if free to move, do so from high potential points to low potential points*. For negative charges the opposite is true: A negative charge placed near plate A will be accelerated toward plate B; that is, *negative charges are accelerated from low potential points to high potential points*. In fact, not only can we say in what direction the charge will accelerate but we can calculate the velocity with which it will reach the other plate, if we know the potential difference between the plates. For this we use the conservation of total mechanical energy, which in this case can be written as

$$E_k(\mathrm{B}) + qV(\mathrm{B}) = E_k(\mathrm{A}) + qV(\mathrm{A}) \qquad (14.20)$$

That is, the sum of the kinetic and potential energies of a charge q at point B is equal to the sum of these energies at point A.

EXAMPLE 14-3

A potential difference of 100 V is established between the two plates of Fig. 14-7, B being the high potential plate. A proton of charge $q = 1.6 \times 10^{-19}$ C is released from plate B. What will be the velocity of the proton when it reaches plate A? The mass of the proton is 1.67×10^{-27} kg.

Solution Because the proton is released with no initial velocity, $E_k(\mathrm{B})$ is zero. From Eq. 14.20, we write

$$E_k(\mathrm{A}) = q[V(\mathrm{B}) - V(\mathrm{A})] = q\Delta V$$

or

$$\frac{1}{2}mv_{\mathrm{A}}^2 = q\Delta V$$

Solving for v_{A}

$$v_{\mathrm{A}} = \sqrt{\frac{2q\Delta V}{m}}$$

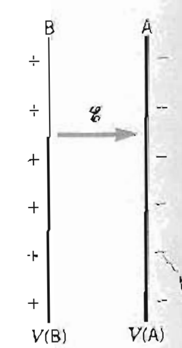


FIGURE 14-7

$$= \sqrt{\frac{2 \times 1.6 \times 10^{-19}\,\text{C} \times 100\,\text{V}}{1.67 \times 10^{-27}\,\text{kg}}}$$

$$= 1.38 \times 10^5\ \text{m/sec}$$

14.5 THE ELECTRON VOLT

A useful unit of energy is the electron volt (eV). Because electric potential difference is the work required per coulomb, then $q\Delta V$ is the energy required to move a charge q through a voltage difference ΔV. The charge of the electron is $q = e = -1.6 \times 10^{-19}$ C (Section 13.4). If an electron is moved through a potential difference of 1 V (1 J/C) the energy change is

$$|e|\Delta V = 1.6 \times 10^{-19}\,\text{C} \times 1\,\text{J/C} = 1.6 \times 10^{-19}\,\text{J}$$

We define 1 electron volt (eV) as 1.6×10^{-19} J. Because the energies of electrons in atoms and solids are of the order of 10^{-19} J, the electron volt is a convenient unit of energy to use in these cases.

Suppose an electron is moved away from a positive charge through a potential difference of 100 V. The electron's potential energy has therefore increased by 100 eV. By energy conservation, if the electron is now released from this point it will acquire a kinetic energy of 100 eV when it arrives back at its starting point.

14.6 ELECTROMOTIVE FORCE

In the electric circuits that we will develop in the next chapter, the symbol ⊣⊢ will be used and labeled with a voltage magnitude, such as 10 V. This symbol represents a battery that is a source of electrical potential energy. A battery is a contained chemical reaction. There are two types, the wet, or rechargeable type used in an automobile, and the dry type, which is used in flashlights. When the chemical reaction in a wet cell is exhausted, it can be recharged a number of times by sending current through it in the reverse direction. When the chemical reaction in a dry cell is exhausted, the battery is "dead." In both types there are two *electrodes* (plates or rods) whose exposed portions are called terminals, the ***anode*** and the ***cathode***. These are suspended in an ionic solution in the wet cell and an ionic gel in the dry cell. The solutions and gels are called *electrolytes*; in wet cells the electrolyte is usually an acid. The anode is made of a material that strongly attracts the positive charges from the electrolyte whereas the cathode is made of a material that has a strong affinity for negative charges. As the anode and cathode attract their respective ions from the solution, they become electrostatically charged to the extent that they cannot attract further ions from the

The car battery is an example of a rechargeable wet cell. The standard D, C, or AA batteries used in flashlights, portable radios and many types of toys are dry cells.

solution. In most electrolytes this balance of forces on the ions, that is, the attraction of the electrode versus the attraction of the solution occurs at around 1.5 V to 2.0 V. This is the approximate voltage between the terminals of a chemical battery. If a wire is connected between the terminals of a battery, charges can flow between the terminals. As the charges on the electrodes decrease in number, the chemical action inside the battery again takes place and charges again migrate from the electrolyte to the plates. In this way the battery maintains a constant potential difference between the plates. This type of potential difference is called an *electromotive force*. This we now know is a misnomer. There is no "force" between the plates, only an electric potential difference. We simply call it *emf*. Higher emfs are obtained by combining cells in series.

The symbol for battery mentioned earlier, ⊣⊢, represents a single cell and is generally used for emfs of 1.5 V or *less*. The large line represents the anode as a source of positive charge, and the small line represents the negative side or the cathode. As mentioned before, early scientists assumed that the charges that flow were the positive ones. To this day we indicate charge flow, or current direction as emanating from the anode ⊣|—. This is called *conventional current*, and its use does not affect the results of common circuit calculations. When a circuit diagram is used in which there is an emf source of several volts the symbol ⊣|||⊢ is used, which represents many batteries in series. The number of these lines depends on space available or persistence of the draftsman, and one should *not* count the batteries drawn to obtain the magnitude of the emf.

14.7 CAPACITANCE

Suppose we connect the terminals of a battery to two parallel metal plates, as in Fig. 14-8. The plate on the left will quickly attain a negative charge of $-q$ and the one on the right a positive charge of $+q$. The plates are characterized by having a charge q, the magnitude of the charge on either of them. It is evident that if the emf of the battery is small, the charge q on the plate will be small and if the emf is large, the charge q will be large. Experiments show that the charge is proportional to the potential difference, ΔV or emf,

$$q \propto V$$

where V actually means ΔV or voltage difference between the two terminals of the battery. We make this relation into an equality by introducing a constant C so that

$$q = CV \tag{14.21}$$

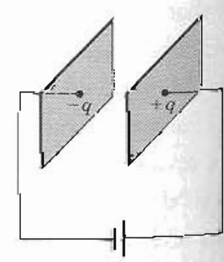


FIGURE 14-8 Two parallel metal plates separated by an insulator, such as air, form a capacitor. When connected to a source of potential difference, the metal plates acquire equal but opposite charges q and $-q$.

The arrangement of such a set of plates as in Fig. 14-8 is called a *capacitor*, and the constant C is called the *capacitance*. The constant has units of

$$C = \frac{q}{V}\frac{\text{(coulomb)}}{\text{(volt)}}$$

which is given the special name of *farad* (abb. F) where

$$1\ \text{farad} = 1\ \text{coulomb/volt}$$

A farad is a very large quantity, and the usual capacitor in an electronic circuit is of the order of microfarads ($1\ \mu F = 10^{-6}$ F) or picofarads ($1\ pF = 10^{-12}$ F). The symbol for a capacitor in an electric circuit is ⊣⊢.

Equation 14.21 represents the charge on a capacitor in a vacuum, and in air there is very little change. Suppose that some nonconducting material, either liquid or solid, is placed between the plates. It is found experimentally that the capacitor will have a higher charge for the same voltage by a factor κ. The material placed between the plates is called a *dielectric* and the factor κ is called the *dielectric constant*. Therefore, Eq. 14.21 is written as

$$q = \kappa CV \tag{14.22}$$

where κ for air or vacuum is unity (1). Some examples of the values of κ are given in Table 14-1.

TABLE 14-1

Material	Dielectric Constant
Vacuum	1
Paper	3.5
Rubber	7

We can also see from Eq. 14.22 that if we wish to maintain a given charge q, less voltage is required with the dielectric present. That is, if it requires V_0 volts to produce a charge q on the capacitor in a vacuum, then when a dielectric is introduced the same charge can be produced by a voltage $V = V_0/\kappa$.

PROBLEMS

14.1 (a) What is the electric field at a point 0.12 m from a point charge of -4×10^{-9} C? (b) What force would an electron experience if it were placed at that point?

14.2 Two equal and opposite charges, $q_1 = 3 \times 10^{-6}$ C and $q_2 = -3 \times 10^{-6}$ C, are held 10 cm apart. (a) What is the electric field at the midpoint of the line joining the two charges? (b) What force would an electron experience if it were placed there?

14.3 The two charges of problem 14.2 are placed on the x axis as shown in Fig. 14-9. What is the electric field at a point on the y axis located at a distance y from the origin?

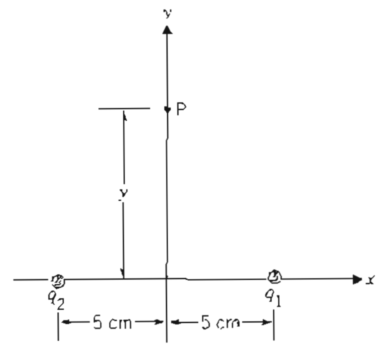


FIGURE 14-9 Problem 14.3.

14.4 Two charges, $q_1 = 7 \times 10^{-8}$ C and $q_2 = -14 \times 10^{-8}$ C, are placed on the x axis as shown in Fig. 14-10. (a) Find the points where the electric field is zero. (b) What is the magnitude and the direction of the electric field at point P with coordinates $x = 0, y = 20$ cm?

Answer: (a) 36.2 cm to the left of q_1,
(b) 1.70×10^4 N/C, $\theta = -28.8°$.

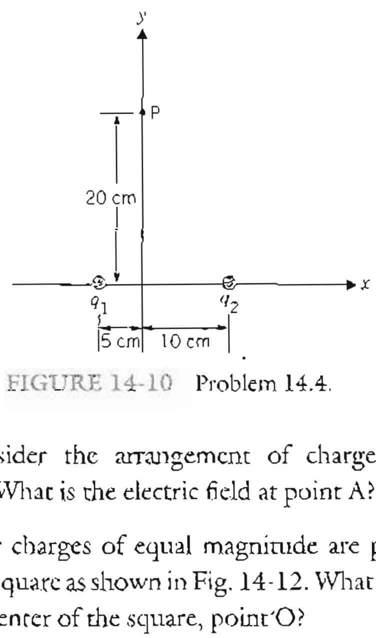


FIGURE 14-10 Problem 14.4.

14.5 Consider the arrangement of charges shown in Fig. 14-11. What is the electric field at point A?

14.6 Four charges of equal magnitude are placed at the corners of a square as shown in Fig. 14-12. What is the electric field at the center of the square, point O?

14.7 Consider the charge configuration of problem 14.6. (a) What is the electric field at point A? (b) What is the electric field at point B?

Answer: (a) 3.77×10^6 N/C directed toward q_2,
(b) 6.19×10^5 N/C directed toward point O.

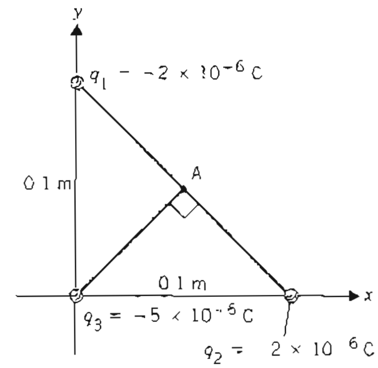


FIGURE 14-11 Problem 14.5.

14.8 Two large parallel plates are separated by a distance of 5 cm. The plates have equal but opposite charges that create an electric field in the region between the plates. An α particle ($q = 3.2 \times 10^{-19}$ C, $m = 6.68 \times 10^{-27}$ kg) is released from the positively charged plate, and it strikes the negatively charged plate 2×10^{-6} sec later. Assuming that the electric field between the plates is uniform and perpendicular to the plates, what is the strength of the electric field?

Answer: 522 N/C.

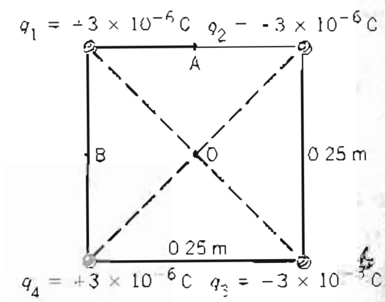


FIGURE 14-12 Problems 14.6 and 14.7.

14.9 An electron is projected with an initial velocity $v_i = 3 \times 10^6$ m/sec in the x direction in the region between two oppositely charged plates (see Fig. 14-13). By the time the electron leaves the region between the plates, it has undergone a vertical deflection of 2 cm. Assume that the electric field between the plates is uniform and perpendicular to the plates and that the electric field outside the region of the plates is zero. (a) What is the strength of the electric field between the plates? (b) At what point y_f on a screen 1 m away from the plates will the electron land?

Answer: (a) 8.19 N/C, (b) 10 cm.

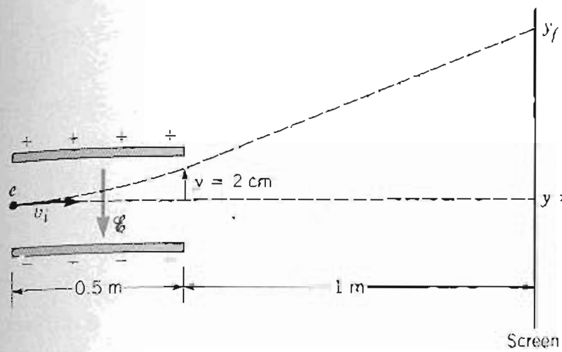


FIGURE 14-13 Problem 14.9.

14.10 A uniform electric field $\mathcal{E} = 500$ N/C exists in the region between two oppositely charged plates (see Fig. 14-14). How much work is done in moving a charge $q = 6 \times 10^{-6}$ C from A to P with constant velocity (a) along path ABP, (b) along path ADP, (c) along the straight line path AP?
Answer: (a) 1.5×10^{-3} J, (b) same as (a), (c) same as (a).

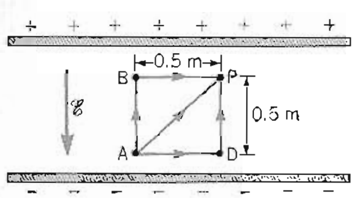


FIGURE 14-14 Problem 14.10.

14.11 The electric field between two parallel plates is uniform and perpendicular to the plates. The potential difference between the plates is 100 V, and the separation between the plates is 1 cm. What is the strength of the electric field between the plates?

Answer: 10^4 N/C.

14.12 What is the potential difference between the plates in problem 14.8?

Answer: 18.3 V.

14.13 A charge $q_1 = 3 \times 10^{-6}$ C is brought from infinity to the origin of a set of coordinate axes. A second charge $q_2 \times 10^{-6}$ C is brought also from infinity to a point with coordinates $x = 5$ cm, $y = 0$ cm. (a) How much work is done in bringing q_1? (b) How much work is done in bringing q_2? (c) What is the potential at $x = 2.5$ cm, $y = 0$ cm? (d) How much work is done in bringing an electron from infinity to the point $x = 2.5$ cm, $y = 0$ cm after q_1 and q_2 have been placed at the locations indicated above?

14.14 (a) What is the potential at point P in Fig. 14.9 of problem 14.3? (b) How much work must be done to move an electron from point P to the origin?

14.15 (a) What is the potential at point O in problem 14.6 (see Fig. 14.13)? (b) What is the potential energy of a charge $q = 1 \times 10^{-6}$ C when it is placed at point O? (c) How much work must be done in bringing q from infinity to point O? (d) How much work must be done to move q from point O to point A?

Answer: (a) 0 V, (b) 0 V, (c) 0 J, (d) 0 J.

14.16 Charged particles are accelerated through a potential difference of 250 V. What will be the kinetic energy in eV if the particle is (a) an electron, (b) a proton, (c) an α particle ($q = +2e$), (d) a gold nucleus ($q = +79e$). e is the magnitude of the charge of the electron and is 1.6×10^{-19} C.

14.17 In a given vacuum tube, an electron is released from the heated filament with zero velocity. It is attracted by the positive plate and arrives at the plate with a velocity of 4×10^6 m/sec. What is the voltage of the plate with respect to the filament? The mass of the electron is 9.1×10^{-31} kg.

14.18 An α particle ($q = +2e$) is shot directly toward a gold nucleus ($q' = +79e$) with a kinetic energy $E_k = 6$ MeV (6×10^6 eV). How close does the α particle get to the gold nucleus? Assume that the gold nucleus remains stationary.

Answer: 3.79×10^{-14} m.

14.19 A particle with charge $q_1 = 4 \times 10^{-6}$ C is held fixed at some point in space. A second particle of mass 20 g and charge $q_2 = -5 \times 10^{-6}$ C is placed 3 cm away from the first particle. What velocity must be given to q_2 so that it will reach infinity with zero velocity?

14.20 Two protons are held fixed 10 cm apart. A third proton is projected from far away with some initial velocity v_i as shown in Fig. 14-15. (a) What is the minimum value of v_i that will allow the proton to reach the midpoint of the line joining the two fixed protons? (b) If the initial velocity is half the value found in part (a), how close to point 0 will the proton get before it stops?

Answer: (a) 3.32 m/sec, (b) 19.4 cm.

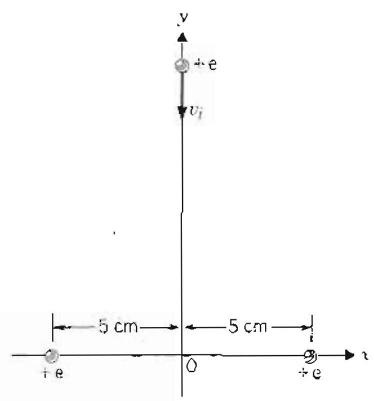


FIGURE 14-15 Problem 14.20.

14.21 An electron is placed midway between two fixed charges, $q_1 = 2.5 \times 10^{-10}$ C and $q_2 = 5 \times 10^{-10}$ C. If the charges are 1 m apart, what is the velocity of the electron when it reaches a point 10 cm from q_2?

14.22 Two particles are placed 1 m apart. Particle 1 has a mass $m_1 = 20$ g and a charge $q_1 = 6 \times 10^{-6}$ C. Particle 2 has a mass $m_2 = 50$ g and a charge $q_2 = -4 \times 10^{-6}$ C. The particles are released from rest simultaneously. (a) What will be the velocities of the particles when they are 0.5 m apart? (b) At what distance from the initial position of particle 1 will the collision occur?

Answer: (a) $v_1 = 3.93$ m/sec, $v_2 = 1.57$ m/sec, (b) 0.714 m.

14.23 An electric dipole consists of two charged particles of mass $m = 300$ g and charge $q_1 = 3 \times 10^{-5}$ C and $q_2 = -3 \times 10^{-5}$ C connected by a rigid rod of negligible weight and length $d = 20$ cm. The dipole is placed in a region where there is a uniform electric field $\mathcal{E} = 5000$ N/C (see Fig. 14-16). (a) What is the torque exerted by the electric field when $\theta = 30°$? (b) How much work must be done to rotate the dipole from the angular position $\theta = 0°$ to $\theta = 90°$? (c) If the dipole is pivoted about its midpoint and is released from the angular position $\theta = 90°$, what will be the angular velocity of the dipole when it swings back to $\theta = 0°$?

Answer: (a) 1.5×10^{-2} N-m, (b) 3×10^{-2} J, (c) 3.16 rad/sec.

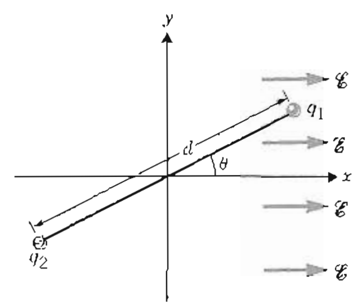


FIGURE 14-16 Problem 14.23.

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15.1 INTRODUCTION

In this chapter we consider the motion of electrons in a conductor (a metal) when there is a voltage difference applied between the ends of the conductor. In more common language, how will the electrons in a metal wire behave if the wire is attached to the two opposite terminals of a battery whose potential difference creates an electric field in the wire? After consideration of the basic behavior, we will treat combinations of wires, batteries, and electrical measuring instruments. It will be necessary to understand the rules for charge flow, a *current*, in circuits in order to follow the arrangement of logic circuits in a computer, which will be treated in Chapter 27. We will limit our discussion mostly to *direct currents*, that is, currents whose magnitude and direction do not change with time.

15.2 MOTION OF CHARGES IN AN ELECTRIC FIELD

We have seen in Chapter 14 that the definition of electric field strength $\mathcal{E}$ is the force per unit positive charge

$$\mathcal{E} = \frac{\mathrm{F}}{q}$$

We may substitute Newton's second law $\mathrm{F} = m\mathrm{a}$

$$\mathcal{E} = \frac{m\mathrm{a}}{q}$$

or

$$\mathrm{a} = \frac{q\mathcal{E}}{m} \qquad (15.1)$$

André Marie Ampère (1775–1836)

Note that in Eq. 15.1 q represents an arbitrary charge. In the case of an electron, $q = e$, where $e = -1.6 \times 10^{-19}$ C and the mass of an electron is $m = 9.1 \times 10^{-31}$ kg. Equation 15.1 will now be written as

$$\mathrm{a} = \frac{-|e|\mathcal{E}}{m} \qquad (15.2)$$

in which the negative sign tells us that the direction of acceleration of an electron is opposite to that of the field direction. Unless there is a specific need to know the direction of motion, we may just use the magnitude of the acceleration in Eq. 15.2. For example, suppose a constant electric field is suddenly applied to a metal. What velocity will the electrons have after traveling a distance s assuming that no scattering (or collisions) occurs over that distance? This is simply a problem from Chapter 3 in which we wish to find a final velocity when the initial velocity is zero and the displacement and the constant acceleration are known. From Eq. 3.11, we write

$$v^2 - v_0^2 = 2ax$$

When

$$v_0 = 0, \quad x = s$$

and

$$a = \frac{e\mathcal{E}}{m}$$

the velocity is

$$v = \sqrt{\frac{2e\mathcal{E}s}{m}}$$

Note that in this expression for v, e stands for the magnitude of the charge of the electron.

15.3 ELECTRIC CURRENT

In the preceding section we saw that an electric field in a conductor can cause charges to undergo accelerated motion. This acceleration is terminated by collisions of the electron with the atoms in the conductor. That is, the motion of an electron in an electric field is a series of short accelerations interrupted by collisions that scatter the electron. It therefore has a random path, although there is a slow net velocity opposite to the field direction, such as illustrated schematically in Fig. 15-1. It is the net velocity of the electrons, called the *drift velocity*, that gives rise to the current, not the brief accelerations.

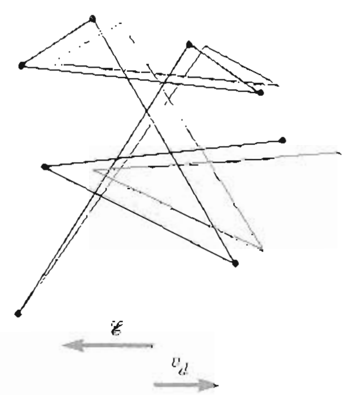


FIGURE 15-1 The random path (black lines) of an electron resulting from collisions with the ions and the effect of an electric field on the path, with a resulting drift velocity (colored lines). [*Source:* David Halliday and Robert Resnick, *Fundamentals of Physics*, 2nd ed Copyright © by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.]

If we stand at a particular plane perpendicular to a wire and count the charge Δq that flows by in time Δt we define this as electric current i, where

$$i = \frac{\Delta q}{\Delta t} \text{C/sec} \qquad (15.3)$$

The definition of the electric current given by Eq. 15.3 holds only if the rate of charge flow is constant. In the general case where i is not constant, we define it as

$$i = \frac{dq}{dt} \qquad (15.3')$$

The concept of electric current is similar to the concept of measuring the current in a river. One measures the quantity of water, in gallons, cubic feet, or such, which flows past a point in a given time.

Because the charge Δq is a scalar quantity, the current i is also a scalar. In the SI units, current is measured in amperes, or amps, after André Ampère (1775–1836).

One ampere (1 A) is equal to one coulomb per second and is a relatively large quantity. Usually in electronic circuits the current is much smaller than 1 A, and we use the milliampere (1 mA $= 10^{-3}$ A) or the microampere (1 μA $= 10^{-6}$ A).

Consider now a cylindrical conductor with a cross-sectional area A as in Fig. 15-2, and let us assume that there are both positive and negative charges, both of which are mobile in the presence of an electric field $\mathcal{E}$ with a vector direction from left to right. Let us further assume that there are N_p positive charges per unit volume with drift velocity of v_p and N_n negative charges with drift velocity of v_n. In time Δt the positive charges will move from left to right a distance of $v_p \Delta t$. Therefore, in time Δt, all the positively charged particles within the shaded region of the cylinder of cross-section A and length $v_p \Delta t$, and only those particles, will flow out of the shaded region of the cylinder to the right. The volume of the shaded cylinder is $A v_p \Delta t$, and the number of positive particles within is the number per unit volume N_p times the volume or $N_p A v_p \Delta t$; if each has a charge q_p, the charge flowing across the right end of the cylinder is

$$\Delta q_p = q_p N_p A v_p \Delta t$$

Substituting this into Eq. 15.3 gives the current resulting from the positively charged particles as

$$i_p = \frac{\Delta q_p}{\Delta t}$$

$$= \frac{q_p N_p A v_p \Delta t}{\Delta t}$$

$$i_p = q_p N_p A v_p \qquad (15.4)$$

In the same way, the negative particles, each with charge q_n, flow from right to left giving rise to a current

$$i_n = q_n N_n A v_n \qquad (15.5)$$

It is seen in Eq. 15.4 that the current i_p of positive charges is to the right in Fig. 15-2 because both the sign of the charge q_p and their drift velocity are positive and, hence, their product is positive. The current i_n resulting from negative charge motion also results in an effective current of positive charges to the right by subtraction of the negative charges. This is seen in Eq. 15.5 in which both the sign of the charge q_n and the sign of the drift velocity v_n are negative and therefore their product is positive: *A flow of negative charges to the left is equivalent to a flow of positive charges to the right.* When both positive and negative charges move, the total current i is the *sum* of these two currents or

$$i = i_p + i_n$$

$$i = A(q_p N_p v_p + q_n N_n v_n) \qquad (15.6)$$

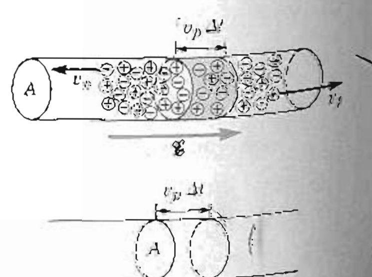


FIGURE 15-2

It is seen by Eq. 15.6 that a measure of electric current does not tell whether the current is being carried by positive or by negative charges or by both.

The vector drift velocity of the positive charge carriers $\mathbf{v}_p$ is in the same vector direction as that of the electric field vector $\mathcal{E}$. The direct current i in a conductor has the same direction as that of the electric field $\mathcal{E}$.

We note that as much charge flows into the cylinder of Fig. 15-2 as flows out. There is no pileup of electric charges in the wire at any point. If there were, then the local electric field would be stronger at that point which would increase the net flow of charge past that point until the charge density at every point in the wire would again be equal. If we connect a wire between the terminals of a battery, it is therefore reasonable to conclude that charge flows at a steady rate throughout the wire.

It is often convenient to introduce a quantity called the current density, with symbol J, which is the current per unit cross-sectional area. This eliminates the area A from Eq. 15.6; thus

$$J = \frac{i}{A}\,\text{A/m}^2(\text{amp/m}^2) \tag{15.7}$$

EXAMPLE 15-1

Suppose a copper wire carries 10 A (amps) of current and has a cross-section of 10^{-6} m^2. As will be seen later, each atom of copper contributes one electron that is free to move, so the electron carrier density N_n is about the same as the density of atoms, which is about 7×10^{28} atoms per m^3 (see problem 9.2). The charge on an electron is -1.6×10^{-19} C. (a) What is the drift velocity v_n of the electrons? (b) How long would it take an electron to move from one terminal of a battery to the other if this wire were 1 m long?

Solution

- (a) $i = A q_n N_n v_n$

$$v_n = \frac{i}{A q_n N_n}$$

$$= \frac{10\,\text{A}}{10^{-6}\,\text{m}^2 \times 1.6 \times 10^{-19}\,\text{C} \times 7 \times 10^{28}\,\text{m}^{-3}}$$

$$= 9 \times 10^{-4}\,\text{m/sec}$$

- (b) $t = \dfrac{x}{v_n} = \dfrac{1\,\text{m}}{9 \times 10^{-4}\,\text{m/sec}} = 1.1 \times 10^3\,\text{sec} = 18\,\text{min}$

So the actual drift velocity of a given electron is very small. However, when we turn on a light switch, the lamp will light almost immediately regardless of the distance from switch to lamp. The reason is that the speed of propagation of the electric field along the wire is that of the speed of light in the wire. Therefore, all electrons are acted on almost simultaneously by the electric field and begin to drift.

15.4 RESISTANCE AND RESISTIVITY

We have seen that the electric field $\mathcal{E}$ is the force per unit charge and, hence, the field causes the charges to be accelerated. The collisions with atoms scatter the charges, which are then accelerated again. This accelerating-scattering process gives rise to a net drift velocity, resulting in an electrical current in the direction of the field $\mathcal{E}$. Experiment shows that in many cases the electric current i, hence the current density J, are proportional to $\mathcal{E}$.

$$J \propto \mathcal{E}$$

We can change this proportionality to an equality by introducing a quantity ρ, called the *electrical resistivity*

$$\mathcal{E} = \rho J \tag{15.8}$$

This resistivity is a property of a given material and is independent of its shape. The resistivity was found to be a constant for a given metal at a given temperature by George Ohm (1789–1854); Eq. 15.8 is called Ohm's law. A material obeying Ohm's is called an *ohmic conductor*, that is, one with a linear relation between current density and electric field. A material with a nonlinear relationship is called a *nonohmic conductor*. For example, in Chapter 26, we will see that a linear dependence does not hold in the case of a circuit element called the diode.

From Eq. 15.8 the units of ρ may be determined

$$\rho = \frac{\mathcal{E}(\text{N/C})}{J(\text{C/sec-m}^2)} = \frac{\mathcal{E}}{J}\left(\frac{\text{N-sec-m}^2}{\text{C}^2}\right)$$

This is such a cumbersome unit that it is shortened to Ω-m (ohm meter).

Some typical values of ρ for conductors and insulators are given in Table 15-1.

TABLE 15-1

Material	ρ (Ω-m) at room temperature
Silver	1.5×10^{-8}
Copper	1.7×10^{-8}
Aluminum	2.7×10^{-8}
Glass	$\sim 10^{+11}$
Teflon	$\sim 10^{+14}$
Dry wood	$\sim 10^{+11}$

George Ohm (1787–1854).

It is seen that differences in resistivity between insulators and conductors can be as great as 22 orders of magnitude (powers of 10). One of the early successes of the theory of solids was to explain this. We will develop this theory in Chapters 23 and 24.

It is sometimes convenient to use the reciprocal of the resistivity; this is called the *conductivity* and has the symbol σ. Equation 15.8 may be written as

$$J = \sigma \mathcal{E} \tag{15.9}$$

where $\sigma = 1/\rho$.

Suppose we have a given metal wire with cross section A, length l, and resistivity ρ with an applied electric field $\mathcal{E}$. The wire is shown schematically in Fig. 15-3. We can use Eq. 14.19 to relate the electric field inside the conductor to the potential difference between the two ends of the conductor, points 1 and 2.

$$\Delta V = V_1 - V_2 = \int_{s_1}^{s_2} \mathcal{E} \cdot ds \tag{14.19}$$

If the electric field inside the conductor is uniform, the integral becomes $\mathcal{E}l$, and

$$\Delta V = \mathcal{E}l$$

where

$$l = s_2 - s_1$$

or

$$\mathcal{E} = \frac{\Delta V}{l} \tag{15.10}$$

Substitution of Eq. 15.10 for $\mathcal{E}$ and Eq. 15.7 for J in Eq. 15.8 yields

$$\Delta V = i\frac{\rho l}{A}$$

which is written

$$V = iR \tag{15.11}$$

where V actually means ΔV or voltage difference between the two ends of the wire. This equation (15.11) is also commonly called *Ohm's law*. In this equation $R = \rho l/A$ and is called the *resistance* of the wire and has units of Ω (ohms). It is seen that the longer the wire the more resistance it has to the current, but the larger its cross-sectional area, the less resistance it has. The analogy to water flowing through a pipe is useful. Voltage will be the equivalent of the difference in water pressure and current that of volume per second of flowing water.

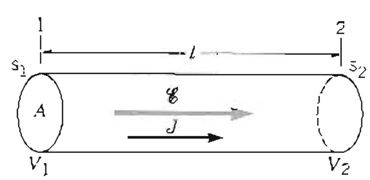


FIGURE 15-3

We should note an important fact about the direction of the current through a resistance. At the beginning of this chapter, Section 15.3, we indicated that the current has the same direction as that of the electric field. In Chapter 14, Section 14.4, we demonstrated that the electric field is directed from high potential points to low potential points. We therefore conclude that *the current in a resistance is from its high potential side to its low potential side*.

15.5 RESISTANCES IN SERIES AND PARALLEL

In this and succeeding sections we will study some simple electrical circuits. Resistances will be drawn with the symbol and emf sources will be drawn as for a small emf and for a larger emf. In all cases, the large line represents the positive, or higher potential, side of the emf. A simple direct current (dc) circuit with one emf source and one resistance is drawn as in Fig. 15-4, where the arrow represents the direction of the current.

In Fig. 15-4 and in the following circuit diagrams the emf sources are labeled with the letter V and the voltage magnitude is given. This is the conventional labeling, but it does not conform well to the definitions. If one measures the potential difference between the terminals of a battery, this is ΔV but, as already mentioned, it is customary to refer to a voltage difference simply as the "voltage." We have also seen that the accepted name for such a voltage source is emf. Confusion in terminology often arises for the student when the form of Ohm's law Eq. 15.11 is used. The V in this equation is the potential, or voltage drop across a resistance R when a current i passes through it. If one takes a meter that measures electric potential difference, called a *voltmeter*, it will read a voltage V when the probes are placed on opposite sides of the resistance. The concept of this voltage difference, commonly called a *voltage drop*, across a resistance should not be confused with the potential difference across the emf source.

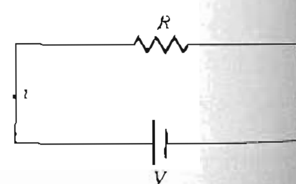


FIGURE 15-4

Suppose that we replace the single resistance of Fig. 15-4 with three resistances (called *resistors*) of different values R_1, R_2, and R_3, as in Fig. 15-5*a*. We assume in this type of calculation that the connecting wires have zero resistance. Therefore, the electric potential at point A is the same as that at the left side of the battery, and that at point D is the same as the right side of the battery. The same current must pass through each of these resistances as that which passes between points A and D. This combination is therefore called *series* resistances because the current passes through each sequentially. By Ohm's law (Eq. 15.11) we may write the voltage drop across each resistance as

$$V_{AB} = iR_1, \quad V_{BC} = iR_2, \quad V_{CD} = iR_3$$

and, because the sum of these voltage drops must equal the potential difference between A and D, V_{AD}, which is the emf of the battery, we conclude that

$$V = V_{AB} + V_{BC} + V_{CD}$$

$$= iR_1 + iR_2 + iR_3 = i(R_1 + R_2 + R_3)$$

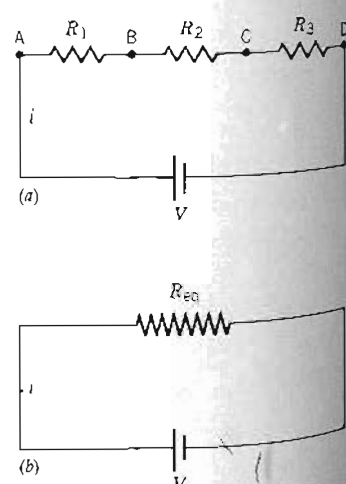


FIGURE 15-5 (*a*) Three resistors in series. (*b*) Equivalent resistor R_{eq}.

$$V = iR_{eq}$$

where R_{eq} is the equivalent resistance of the three. That is, the same current would flow through the circuit if the three resistances were replaced by a single one (Fig. 15-5*b*) with magnitude

$$R_{eq} = R_1 + R_2 + R_3 \qquad \text{(series)} \qquad (15.12)$$

It is obvious that the generalization of Eq. 15.12, namely, $R_{eq} = \sum_i R_i$ will be true regardless of the number of resistances in series.

EXAMPLE 15-2

Suppose in Fig. 15-5 the voltage $V = 1.5\,\text{V}$ and the resistances are $R_1 = 5\,\Omega$, $R_2 = 10\,\Omega$, and $R_3 = 15\,\Omega$. What are the voltages V_{AB}, V_{BC}, and V_{CD}?

Solution First we find the current through the resistors by replacing the individual resistances with a simple equivalent resistance.

$$V = iR_{eq} = i(R_1 + R_2 + R_3)$$

$$i = \frac{1.5\,\text{V}}{(5 + 10 + 15)\Omega} = 0.05\,\text{A} = 50\,\text{mA}$$

Then, applying Ohm's law to each resistance

$$V_{AB} = iR_1 = 0.05\,\text{A} \times 5\,\Omega = 0.25\,\text{V}$$

$$V_{BC} = iR_2 = 0.05\,\text{A} \times 10\,\Omega = 0.50\,\text{V}$$

$$V_{CD} = iR_3 = 0.05\,\text{A} \times 15\,\Omega = 0.75\,\text{V}$$

$$\text{sum} = V_{AD} = 1.5\,\text{V}$$

Suppose we now arrange these resistances in *parallel*, as in Fig. 15-6*a*. As stated previously, we assume the resistance of connecting wires to be negligible and, for clarity, we redraw the circuit of Fig. 15-6*a* in the form of Fig. 15-6*b*. Because all connecting wires are considered to have zero resistance, there can be no voltage drop across them. Therefore, the left side of each resistance is at the same potential and the right side is at the same potential; hence, the same voltage drop V must occur across each. We further note that although the current through each resistance may be different, the sum of the individual currents must equal the current that flows through the wire connecting them to the battery because charge must be conserved. Thus,

$$i = i_1 + i_2 + i_3 \qquad \text{(parallel)} \qquad (15.13)$$

Because each resistance has the same voltage drop v across it, we may write Ohm's law for each

$$V = i_1R_1, \quad V = i_2R_2, \quad V = i_3R_3$$

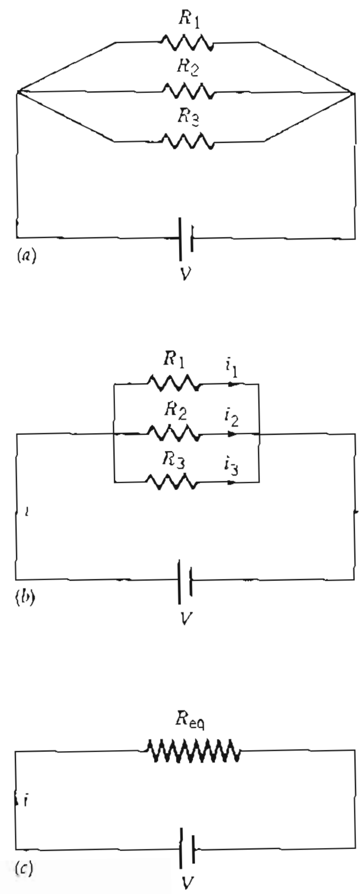


FIGURE 15-6 (*a*) Three resistors in parallel. (*b*) Conventional form of drawing resistors in parallel. (*c*) Equivalent resistor R_{eq}.

and

$$i_1 = \frac{V}{R_1}, \quad i_2 = \frac{V}{R_2}, \quad i_3 = \frac{V}{R_3}$$

Substituting these into Eq. 15.13 gives

$$i = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

$$= V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

or

$$V = \frac{i}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)}$$

$$= \frac{i}{\frac{1}{R_{eq}}} = iR_{eq}$$

where

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \qquad \text{(parallel)} \qquad (15.14)$$

We see that any number of parallel resistors can be replaced with an equivalent resistor by a generalization of the relation of Eq. 15.14. The equivalent circuit looks like Fig. 15-6*c*, where R_{eq} is given by Eq. 15.14.

EXAMPLE 15-3

Suppose two resistors, $R_1 = 5\ \Omega$ and $R_2 = 10\ \Omega$, are connected in parallel to a 1.5-V battery as in Fig. 15-7*a*. (a) What is the current through each? (b) What is the total current in the circuit?

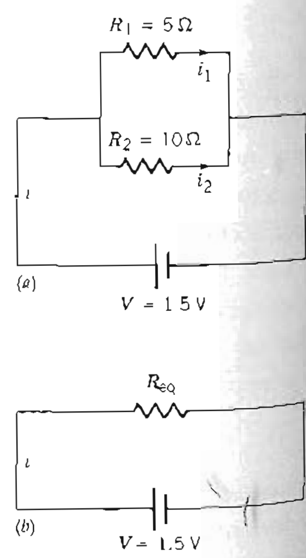


FIGURE 15-7 Example 15-3.

Solution

- (a) Using Ohm's law (Eq. 15.11)

$$V = i_1R_1, \quad V = i_2R_2$$

$$i_1 = \frac{V}{R_1} = \frac{1.5\,\text{V}}{5\,\Omega} = 0.3\,\text{A} = 300\,\text{mA},$$

$$i_2 = \frac{V}{R_2} = \frac{1.5\,\text{V}}{10\,\Omega} = 0.15\,\text{A} = 150\,\text{mA},$$

- (b) $i = i_1 + i_2 = 300\,\text{mA} + 150\,\text{mA} = 450\,\text{mA}$

We may check this answer by solving the equivalent circuit, Fig. 15-7*b*.

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{5\,\Omega} + \frac{1}{10\,\Omega} = 0.2\,\Omega^{-1} + 0.1\,\Omega^{-1} = 0.3\,\Omega^{-1}$$

or

$$R_{eq} = \frac{1}{0.3\,\Omega^{-1}} = 3.33\,\Omega$$

$$i = \frac{V}{R_{eq}} = \frac{1.5\,\text{V}}{3.33\,\Omega} = 0.45\,\text{A} = 450\,\text{mA}$$

EXAMPLE 15-4

Three resistors are connected in a combination of series and parallel as in Fig. 15-8*a*. What is the current through each?

Solution First we find $R_{eq(p)}$ for the parallel combination

$$\frac{1}{R_{eq(p)}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2\,\Omega} + \frac{1}{4\,\Omega} = 0.5\,\Omega^{-1} + 0.25\,\Omega^{-1} = 0.75\,\Omega^{-1}$$

$$R_{eq(p)} = 1.33\,\Omega$$

We then have the equivalent circuit, Fig. 15-8*b*. We now find the equivalent series resistance $R_{eq(s)}$

$$R_{eq(s)} = R_{eq(p)} + R_3 = 1.33\,\Omega + 3\,\Omega = 4.33\,\Omega$$

We now have the simpler equivalent circuit of Fig. 15-8*c*. The current is given by Ohm's law

$$i = \frac{V}{R_{eq(s)}} = \frac{1.5\,\text{V}}{4.33\,\Omega} = 0.35\,\text{A} = 350\,\text{mA}$$

We may return to Fig. 15-8*b* and note that we have already solved part of the problem because all this current flows through R_3, hence, $i_3 = 350\,\text{mA}$. We may find the voltage drop across the parallel combination by use of Ohm's law

$$V_{(p)} = iR_{eq(p)} = 350\,\text{mA} \times 1.33\,\Omega = 0.47\,\text{V}$$

The current through each of the parallel resistors is then

$$i_1 = \frac{V_{(p)}}{R_1} = \frac{0.47\,\text{V}}{2\,\Omega} = 0.235\,\text{A} = 235\,\text{mA}$$

$$i_2 = \frac{V_{(p)}}{R_2} = \frac{0.47\,\text{V}}{4\,\Omega} = 0.118\,\text{A} = 118\,\text{mA}$$

And, except for the rounding-off error, $i_1 + i_2 = i_3 = i$

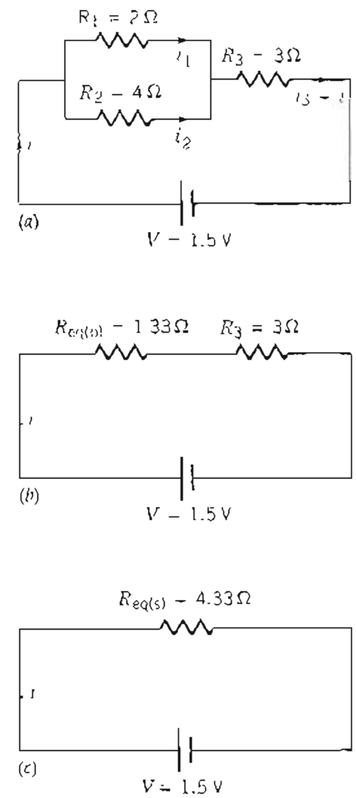


FIGURE 15-8 Example 15-4.

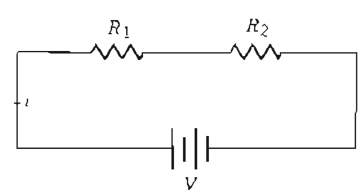


FIGURE 15-9

You will have noticed that when there are two resistors in series or in parallel, a method of ratios exists as a quicker way of solution. Consider first a series circuit as in Fig. 15-9. The current through R_1 is the same as that through R_2, and by Ohm's law

$$i = \frac{V_1}{R_1}, \quad i = \frac{V_2}{R_2}$$

where V_1 and V_2 are the voltage drops across R_1 and R_2, respectively. Equating the i's gives

$$\frac{V_1}{R_1} = \frac{V_2}{R_2}$$

or

$$\frac{V_1}{V_2} = \frac{R_1}{R_2} \qquad \text{(series)} \qquad (15.15)$$

so that *in a series circuit the ratio of the voltage drops is equal to the ratio of the resistances.*

A different ratio can be written for parallel resistors (see Fig. 15-10). In this situation we recall that the voltage across each is the same. From Ohm's law

$$V_1 = i_1R_1, \quad V_2 = i_2R_2$$

Equating V_1 and V_2 gives

$$i_1R_1 = i_2R_2$$

or

$$\frac{i_1}{i_2} = \frac{R_2}{R_1} \qquad \text{(parallel)} \qquad (15.16)$$

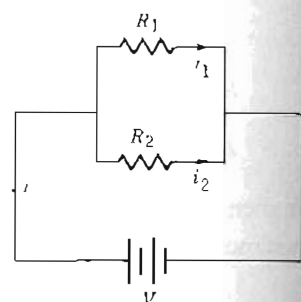
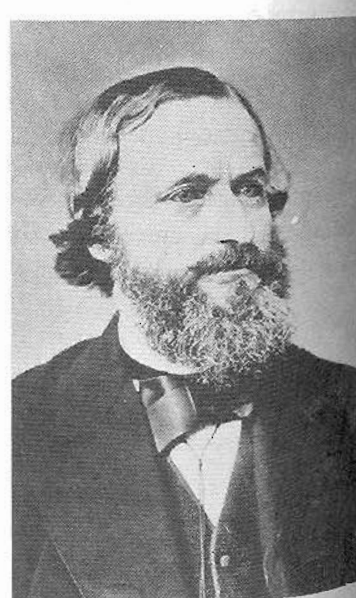


FIGURE 15-10

where we see that *in a parallel circuit the ratio of the currents through each resistor is inversely proportional to the resistances.* We might have expected this result from the understanding that resistance impedes the flow of current; hence, the larger the resistance the lower the current.

15.6 KIRCHHOFF'S RULES


Gustav R. Kirchhoff (1824–1887).

Not all electrical circuits can be reduced to simple series or parallel combinations. Two fundamental rules were established by G. R. Kirchhoff (1824–1887) that aid in the solution of electrical networks.

1. The algebraic sum of currents *toward* any branch point is zero.
2. The algebraic sum of all potential changes in a closed loop is zero.

We will consider rule (1) first. As we have stated earlier, charge cannot accumulate (or be depleted) in a dc circuit: If it did, there would be a larger (or smaller) electric field at that region which would exert a larger (or smaller) force and thereby redistribute the charge evenly. Therefore, at any branch point in a circuit, whatever charge flows in must flow out. This is seen in Figs. 15-11*a* and *b*. An analogy would be a series of hoses for water irrigation. At a branch of hoses, whatever water flows in must flow out.

The current equation for Fig. 15-11*a*, $i_1 = i_2 + i_3$, is an obvious application of the first rule. The rule can also be used for the circuit of Fig. 15-11*b*. Consider branch point A. The current equation would be $i_2 = i_3 + i_4$ *if* the currents have the directions indicated by the arrows. We do know that current i_2 has the direction indicated by the arrow. But we do not know if the arrow is the correct direction for i_3 since we do not know if the potential at point A is higher or lower than that at point B. We therefore *assume* a direction and maintain that assumption in formulating other equations. When we finally solve the circuit equations, if i_3 is positive our assumption is correct; if it is negative, then the current i_3 is in the direction opposite to our assumption. To illustrate the consistency of following the original assumption, rule (1) applied to branch point B would be $i_1 + i_3 = i_5$.

Let us consider rule (2). This rule is a statement of the conservation of energy. In a circuit there may be potential differences associated with emf sources present as well as voltage drops associated with resistors. If we mentally start at a point in the circuit, go around any closed loop in either direction adding algebraically all the changes in potential and then return to the starting point, the potential of that point must be the same as when we started; that is, the sum of all the potential changes (increases and decreases) considered in our mental trip must add up to zero.

In applying rule (2), it is useful to follow certain guidelines that will prevent errors in the signs of the potential changes.

(a) As indicated in connection with rule (1), we first assume a direction for the current through each branch of the circuit.

(b) We then choose any closed loop in the circuit and designate the direction (clockwise or counterclockwise) in which we wish to mentally traverse it.

(c) We now go around the loop in the chosen direction adding algebraically all the potential changes and setting the sum equal to zero.

When we meet an emf source, its voltage V is taken as positive if we cross the source from the negative (low potential) side to the positive (high potential) side. The reason for taking V as positive is that in going from the negative side of the source to the positive one, the change in potential represents an *increase* in electric potential. If the source is crossed from positive to negative, its voltage is taken as negative because the electric potential has decreased. Let us now consider what to do when we meet a

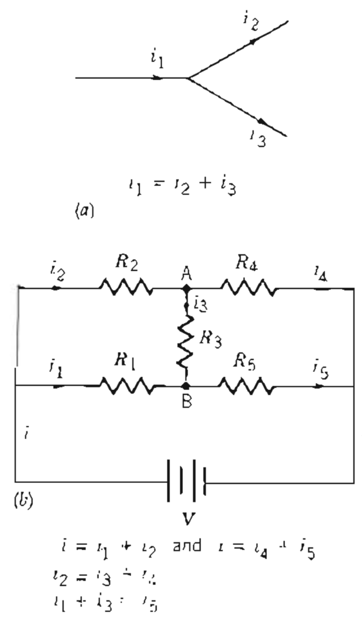


FIGURE 15-11 Kirchhoff's rule for current at a branch point.

resistor. Earlier, we indicated that in a resistor the current goes from the high potential side of the resistor to the low potential side and that the potential drop between the two sides is iR. Thus, if in our mental trip around the circuit loop we cross a resistor in the same direction as the current, we must take the iR drop as negative because we are going from high to low potential—a decrease. The iR drop is taken as positive if the resistor is traversed in the direction opposite to that of the current.

Now we can apply rule (2) to some simple circuits. Consider the circuit of Fig. 15-12. We choose the current in the clockwise direction (we may just as well choose the opposite direction, although the correctness of our assumption is obvious by inspection in this simple circuit). If we now traverse the loop in the clockwise direction starting at point A, we apply rule (2) and write

$$-iR_1 - iR_2 + V = 0$$

Note that both iR drops are written as negative because both resistors were crossed in the direction of the assumed current. V was taken as positive because the emf source was crossed from the negative to the positive side. We can rewrite the result as

$$V = iR_1 + iR_2$$

which is the result obtained when we discussed resistors in series where rule (2) was implicitly used.

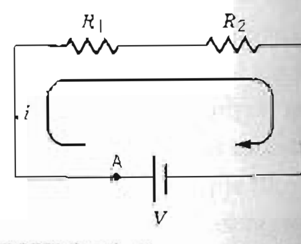


FIGURE 15-12

Let us now consider the slightly more complicated circuit of Fig. 15-13*a*. We may again choose to go around the loop in a clockwise direction starting at point A. From rule (2) we obtain

$$-iR_1 - iR_2 + V_2 + V_1 = 0$$

or

$$V_2 + V_1 = iR_1 + iR_2$$

We see that because the batteries are pointing in the same direction (relative to the direction of the current) the effective voltage of the two emf sources is the sum of the individual voltages, that is, an emf source of voltage $V = V_1 + V_2$ would give rise to the same current. Suppose we reverse the direction of one of the sources, as in Fig. 15-13*b*. We will still *assume* that the current is in the clockwise direction. With such an assumption, and if we traverse the loop in the clockwise direction, we write

$$-iR_1 - iR_2 + V_2 - V_1 = 0$$

Note that V_1 is now taken as negative because in our mental trip the emf V_1 was crossed from the positive to the negative side. The result can be solved for i

$$i = \frac{V_2 - V_1}{R_1 + R_2}$$

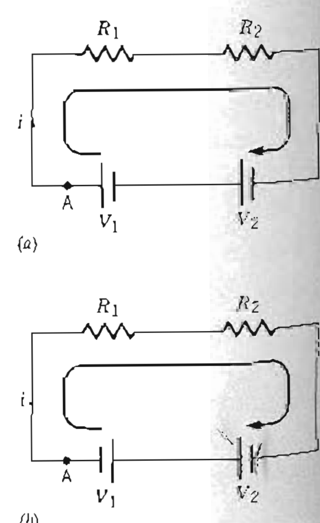


FIGURE 15-13

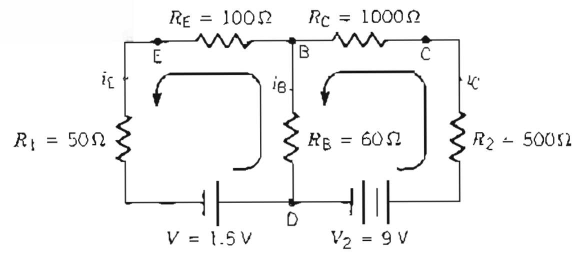


FIGURE 15-14 Example 15-5.

It is clear that if $V_1 > V_2$, i would be negative, indicating that we have assumed the wrong direction for i, that is, i would be counterclockwise.

Let us now use Kirchhoff's rules to solve a circuit with certain similarities to a transistor circuit that we will encounter in a later chapter.

EXAMPLE 15-5

In the circuit of Fig. 15-14, (a) Find the currents i_C, i_E, and i_B and the voltage drop across resistors R_1 and R_2. (b) Find the voltage difference between points C and D and between D and E.

Solution

▪ (a) From the first rule at branch point B

$$i_C + i_B = i_E$$

We then write the second rule for the two loops. For the right-hand loop, if we traverse it in the counterclockwise direction starting at point D, we write

$$V_2 - i_C R_2 - i_C R_C + i_B R_B = 0$$

$$9\text{ V} - i_C 500\ \Omega - i_C 1000\ \Omega + i_B 60\ \Omega = 0$$

$$9\text{ V} - i_C 1500\ \Omega + i_B 60\ \Omega = 0$$

For the left-hand loop, traversing it counterclockwise, we write

$$V_1 - i_B R_B - i_E R_E - i_E R_1 = 0$$

$$1.5\text{ V} - i_B 60\ \Omega - i_E 100\ \Omega - i_E 50\ \Omega = 0$$

$$1.5\text{ V} - i_B 60\ \Omega - i_E 150\ \Omega = 0$$

We now have three equations to be solved simultaneously for i_C, i_E, and i_B. They are

$$i_C + i_B = i_E$$

$$9\,\text{V} - i_C 1500\,\Omega + i_B 60\,\Omega = 0$$

$$1.5\,\text{V} - i_B 60\,\Omega - i_E 150\,\Omega = 0$$

We can use the first equation to eliminate i_B from the last two. $i_B = i_E - i_C$, therefore

$$9\,\text{V} - i_C 1500\,\Omega + (i_E - i_C) 60\,\Omega = 0$$

or

$$9\,\text{V} - i_C 1560\,\Omega + i_E 60\,\Omega = 0 \qquad (15.17)$$

and

$$1.5\,\text{V} - (i_E - i_C) 60\,\Omega - i_E 150\,\Omega = 0$$

or

$$1.5\,\text{V} - i_E 210\,\Omega + i_C 60\,\Omega = 0 \qquad (15.18)$$

Equations 15.17 and 15.18 can be solved for i_E and i_C. Multiplying Eq. 15.17 by 7 and Eq. 15.18 by 2 and adding them together eliminates i_E, that is,

$$7 \times (9\,\text{V} - i_C 1560\,\Omega + i_E 60\,\Omega) = 0$$

$$2 \times (1.5\,\text{V} - i_E 210\,\Omega + i_C 60\,\Omega) = 0$$

$$63\,\text{V} - i_C 10,920\,\Omega + 3\,\text{V} + i_C 120\,\Omega = 0$$

$$i_C = \frac{66\,\text{V}}{10,800\,\Omega} = 6.1 \times 10^{-3}\,\text{A} = 6.1\,\text{mA}$$

We can now solve for i_E using either Eq. 15.17 or 15.18

$$1.5\,\text{V} - i_E 210\,\Omega + i_C 60\,\Omega = 0$$

$$i_E = \frac{1.5\,\text{V} + i_C 60\,\Omega}{210\,\Omega}$$

$$= \frac{1.5\,\text{V} + (6.1 \times 10^{-3}\,\text{A})(60\,\Omega)}{210\,\Omega}$$

$$= 8.9 \times 10^{-3}\,\text{A} = 8.9\,\text{mA}$$

Finally, we can use the result of the first rule, $i_C + i_B = i_E$, to obtain i_B

$$i_B = i_E - i_C = 8.9\,\text{mA} - 6.1\,\text{mA} = 2.8\,\text{mA}$$

The voltage drop across R_1 is

$$V = i_E R_1 = 8.9 \times 10^{-3}\,\text{A} \times 50\,\Omega = 0.45\,\text{V}$$

and the voltage drop across R_2 is

$$V = i_C R_2 = 6.1 \times 10^{-3}\,\text{A} \times 500\,\Omega = 3.1\,\text{V}$$

(b) To find the voltage difference between points D and C, let us follow the current i_C through the circuit elements in the right-hand loop. Start with the potential V_D.

$$V_D + V_2 - i_C R_2 = V_C$$

$$V_C - V_D = V_2 - i_C R_2$$

$$= 9\,\text{V} - 6.1 \times 10^{-3}\,\text{A} \times 500\,\Omega$$

$$= 6\,\text{V}$$

We may do the same with the left-hand loop to find the potential difference between points D and E.

$$V_D - i_B R_B - i_E R_E = V_E$$

$$V_E - V_D = -i_B R_B - i_E R_E$$

$$= -2.8 \times 10^{-3}\,\text{A} \times 60\,\Omega$$

$$-8.9 \times 10^{-3}\,\text{A} \times 100\,\Omega$$

$$= -1.1\,\text{V}$$

The result shows that D is at a higher potential than E.

15.7 AMMETERS AND VOLTMETERS

In order to have an intuitive grasp of the operation of a circuit, it is important to understand how current or voltage could be *measured* at any location within the circuit. The simple meters for current or voltage measurement described in this section were employed for many decades, prior to the era of modern electronics, and illustrate some basic principles of circuit design and analysis.

We will see in the next chapter that electric current passing through a wire produces a magnetic field. If a loop of wire is used then, on the passage of current, one end of the loop becomes the north pole of a magnet and the other end becomes the south pole, as in Fig. 15-15*a*. This will be discussed in more detail in Chapter 16. Many loops can be used (see Fig. 15-15*b*), and each one contributes to the forming of a magnet: The larger the number of loops, the stronger the magnet for a given current. A series of loops forms a coil, and if a tightly wound coil is placed between the poles

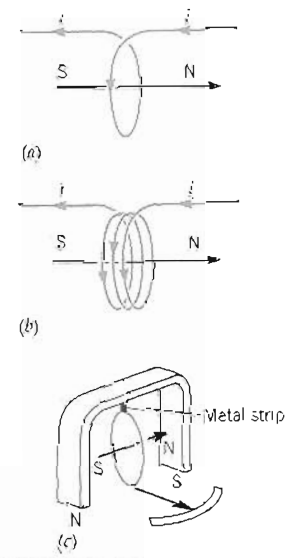


FIGURE 15-15 The basis of a galvanometer.

of a permanent horseshoe magnet as in Fig. 15-15*c* and current passes through it, the induced north pole of the coil will be repelled by the north pole of the permanent magnet. If the coil is suspended by a flexible metal strip (see Fig. 15-15*c*), the twisting (torsion) force of this metal strip acts as a spring and will oppose the rotation of the coil, causing it to return to its initial position at zero current. Thus, depending on the strength of this metal strip and the other design parameters of the mechanism, a full-scale deflection of the instrument needle can be established for a given amount of current through the coil. This instrument is called a *galvanometer*. The current for full-scale deflection is called the *current rating* of a meter. Because these instruments are electrically and mechanically delicate, a common current rating is 0.1 mA (10^{-4} A).

Used by itself, such a meter could only measure currents from 0 to 0.1 mA when placed in series in the circuit. To extend the range of the meter, a lower resistance, called a *shunt*, is placed in parallel with the meter. Figure 15-16 shows these situations in which the meter could measure full scale for different currents in the line. Meters used for the measurement of current though a circuit are known by the general name of *ammeters* (meters to measure amps of current). The resistance of the coil R_c in the galvanometer is also specified by the manufacturer, in addition to the current rating. A typical value might be 1000Ω, or one kilohm, kΩ. From Ohm's law the voltage drop across the galvanometer in all cases of Fig. 15-16 must be

$$V = iR_c = 10^{-4}\,\text{A} \times 10^3\,\Omega = 0.1\,\text{V}$$

and this must also be the voltage drop across the shunt because it is in parallel with the meter. With the use of Ohm's law we may calculate the value of the shunt resistance. In Fig. 15-16*b*

$$R_s = \frac{V}{i_s} = \frac{0.1\,\text{V}}{9.9 \times 10^{-3}\,\text{A}} = 10.1\,\Omega$$

In Fig. 15-16*c*

$$R_s = \frac{V}{i_s} = \frac{0.1\,\text{V}}{99.9 \times 10^{-3}\,\text{A}} = 1.001\,\Omega$$

Many test meters have an external switch that changes the scale of the ammeter. This switch disconnects one shunt and introduces one of a different resistance into the circuit so that the same meter is used for different ranges of current.

An instrument to measure the voltage difference between two points in a circuit, say, two sides of a resistor, is called a *voltmeter* and can be made from a similar galvanometer. However, we do not want such a voltmeter to disturb the current flow through the resistor because such a change would alter the *iR* voltage drop. The ideal instrument would be one that had infinite resistance. However, the galvanometer requires that current pass through it to obtain a measurement. We will continue to use the galvanometer previously discussed, which requires 100 μA for full-scale deflection and has an internal resistance of 1 kΩ (1000 Ω). As shown before, this meter will

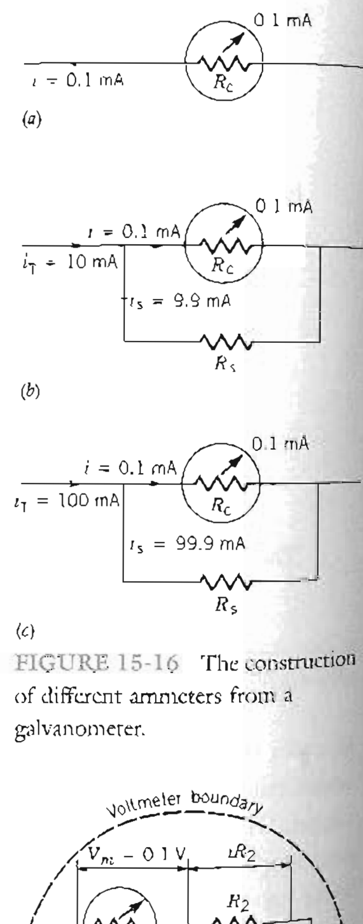


FIGURE 15-16 The construction of different ammeters from a galvanometer.

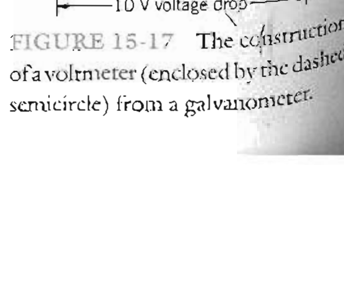


FIGURE 15-17 The construction of a voltmeter (enclosed by the dashed semicircle) from a galvanometer.

read full scale when the voltage difference across it is $V = 10^{-4}\,\text{A} \times 10^3\,\Omega = 0.1\,\text{V}$. Suppose we wish to extend the range of the galvanometer to 10 V full scale. We would connect a resistor R_2 in series with the galvanometer so that the potential drop across the galvanometer is still 0.1 V, as in Fig. 15-17. To find the value of R_2, we note that the voltage drop across the galvanometer $V_m = 0.1\,\text{V}$, plus the voltage drop across R_2, iR_2, must be 10 V, that is,

$$0.1\,\text{V} + 10^{-4}\,\text{A} \times R_2 = 10\,\text{V}$$

$$R_2 = \frac{9.9\,\text{V}}{10^{-4}\,\text{A}} = 9.9 \times 10^4\,\Omega$$

Similarly, if the meter is to read full scale across a voltage drop of 100 V, the drop across R_2 must be $100 - 0.1 = 99.9\,\text{V}$ and the value of R_2 must be

$$R_2 = \frac{99.9\,\text{V}}{10^{-4}\,\text{A}} = 9.99 \times 10^5\,\Omega$$

Again, an external switch on a meter connects different values of series resistances so that full-scale deflection of the meter may indicate different maximum voltages.

15.8 POWER DISSIPATION BY RESISTORS

We saw at the end of Chapter 6 that in an elastic collision between an electron and an atom, very little energy is transferred to the atom—most of the kinetic energy is retained by the electron in its recoil (bouncing off the atom). However small, some energy is lost by the electron to the atom. This is the situation in a metal when electrons, accelerated by the electric field, collide with the atoms. Because many collisions are taking place, each small energy loss adds to a considerable amount. The kinetic energy transferred to the atoms per unit time represents an energy loss per unit time by the electrons, which is a power loss. We have seen in Chapter 9 that temperature is a measure of the average kinetic energy of the atoms (or molecules) of a system. Therefore, we expect any conductor to heat up when an electric current is passed through it. We see this phenomenon daily in electric heaters, ovens, and light bulbs.

The calculation of power dissipation P in an electrical resistor R as a result of the passage of a current i can immediately be found by considering Fig. 15-18. Let V_A and V_B represent the potentials of points A and B, respectively, and V_{AB} the potential difference. The change in potential energy of a charge Δq entering at A and leaving at B is

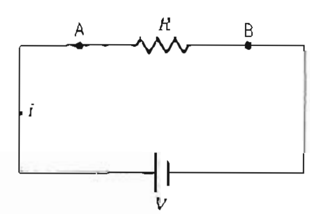


FIGURE 15-18

$$\Delta E_p = \Delta q(V_B - V_A)$$

This represents an energy loss because V_A is greater than V_B. In a given time Δt the amount of charge involved is $\Delta q = i\Delta t$, so

$$\Delta E_p = V_{AB} i \Delta t$$

The power dissipated in the resistor is $P_{AB} = \Delta E_p/\Delta t$, or

$$P_{AB} = V_{AB} i$$

We have written the voltage with the subscript AB to denote that if there is more than one resistor in a circuit, the power lost in each is the product of the voltage across each and current flowing through each. With this point in mind, we may write in more general form

$$P = Vi \tag{15.19}$$

Two other forms may be obtained by substitution of Ohm's law, $V = iR$. These are

$$P = i^2 R \tag{15.20}$$

and

$$P = \frac{V^2}{R} \tag{15.21}$$

We have shown in Chapter 5 that the unit of power is the watt or J/sec. The consistency in electrical definitions is readily seen from Eq. 15.19. The unit of voltage is one joule per coulomb (one volt) and the unit of current is one coulomb per second (one ampere). Therefore the unit of power is

$$P = Vi = (1\,\text{JC}^{-1})(1\,\text{C sec}^{-1}) = 1\,\text{J/sec} = 1\,\text{W (watt)}$$

15.9 CHARGING A CAPACITOR—RC CIRCUITS

Thus far we have limited our discussion to cases where the current is constant with time, that is, direct current. In this section we consider a circuit where the current varies with time. This circuit plays an important role in the operation of computer clocks, which will be presented in Chapter 27.

In Chapter 14 (Section 14.7) we saw that when a capacitor is connected to the terminals of a battery, the plate of the capacitor connected to the positive side of the battery acquires a positive charge $q = CV$ (Eq. 14.21) and the other plate an equal but negative charge $-q$. One question that we may ask is: How long does it take for the charges to appear on the plates of the capacitor? Obviously, because resistors determine the current (that is, the rate of charge flow) in the circuit, this will depend on the resistance that is in the circuit.

Let us consider a circuit where a resistor R (the resistance of the connecting wires is assumed to be negligible) and a capacitor C are connected in series by means of switch S to a battery of emf V as in Fig. 15-19*a*. The initial condition is that when the switch is open there is no charge on the capacitor. When the switch is closed, a current is set up in the circuit and the capacitor will begin to charge (see Fig. 15-19*b*).

Let q be the charge on the capacitor at some time t after the switch is closed and let i be the current through the resistor at the same instant. V_{AD} is the voltage across the terminals of the battery, that is, $V_{AD} = V$, V_{AB} is the voltage drop iR across the resistor, and V_{BD} is the potential difference $\frac{q}{C}$ across the plates of the capacitor. We therefore write

$$V_{AD} = V_{AB} + V_{BD}$$

or

$$V = iR + \frac{q}{C} \tag{15.22}$$

By definition i is the rate at which charges flow through the resistor and, because these charges cannot cross the gap between the capacitor plates, this rate represents the rate at which the charge on the capacitor is increasing, that is, $i = \frac{dq}{dt}$. Equation 15.22 becomes

$$V = R\frac{dq}{dt} + \frac{q}{C} \tag{15.23}$$

Dividing both sides of Eq. 15.23 by R and rearranging terms we have

$$\frac{V}{R} - \frac{q}{RC} = \frac{dq}{dt}$$

or

$$dt = \frac{dt}{\frac{V}{R} - \frac{q}{RC}} \tag{15.24}$$

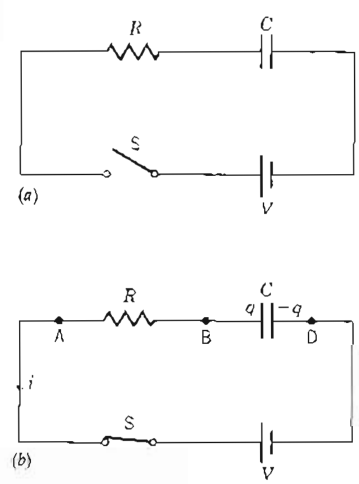


FIGURE 15-19

We multiply both sides of Eq. 15.24 by $-1/RC$ and integrate with the limits $q = 0$ when $t = 0$ and charge q is on the plates at time t.

$$-\frac{1}{RC}\int_0^t dt = \int_0^q \frac{-\frac{1}{RC}dq}{\frac{V}{R} - \frac{q}{RC}}$$

$$-\frac{t}{RC} = \ln\left(\frac{V}{R} - \frac{q}{RC}\right)\Big|_0^q$$

Substituting the limits of integration we get

$$-\frac{t}{RC} = \ln\left(\frac{V}{R} - \frac{q}{RC}\right) - \ln\left(\frac{V}{R}\right) \tag{15.25}$$

Because $\ln x - \ln y = \ln \frac{x}{y}$, Eq. 15.25 becomes

$$-\frac{t}{RC} = \ln\left(1 - \frac{q}{VC}\right)$$

Taking the antilog of both sides we obtain

$$e^{-t/RC} = 1 - \frac{q}{CV}$$

Solving for q, we get

$$q = CV(1 - e^{-t/RC}) \qquad (15.26)$$

Let us analyze Eq. 15.26. At $t = 0$, $q = CV\,(1 - e^{-0}) = CV\,(1 - 1) = 0$. This agrees with the fact that at $t = 0$ (when the switch was closed) the capacitor was uncharged. As t increases, the exponential term in the parenthesis decreases and consequently q increases (the capacitor is being charged). As $t \to \infty$, $e^{-t/RC} \to 0$ and $q \to CV$, the ultimate charge on the capacitor. Although it takes an infinite amount of time to fully charge the capacitor, it takes a finite amount of time to get very close to the final value $q = CV$. Moreover, this time is determined by the product RC, which is called the time constant of the circuit. For example, when $t = RC$, $q = CV\,(1-e^{-1}) = 0.63\,CV$; when $t = 4\,RC$, $q = CV\,(1 - e^{-4}) = 0.98\,CV$. We see that after a few time constants, q is very close to its ultimate value. A plot of q versus t is shown in Fig. 15-20.

The consistency of the electrical definitions of R and C can be readily verified in Eq. 15.26. From Ohm's law,

$$R = \frac{V}{i}\left(\frac{\text{volts}}{\text{amps}}\right) \text{ or } \left(\frac{\text{volts}}{\text{coul/sec}}\right).$$

Similarly, from Eq. 14.21

$$C = \frac{q}{V}\left(\frac{\text{coul}}{\text{volts}}\right)$$

therefore, the units of RC become

$$RC\left(\frac{\text{volts}}{\text{coul/sec}}\,\frac{\text{C}}{\text{volts}}\right) \text{ or (seconds).}$$

The mathematical solution of the circuit equation shows that the larger the value of the resistor R and of the capacitor C, the longer it will take to charge it. It is not difficult to understand the physical reason for this result. The larger R, the smaller the current through the circuit at any one time, hence the smaller the rate at which the capacitor is being charged. Similarly, the larger C, the more charge it can store for a given voltage and, obviously, the longer it will take to charge it.

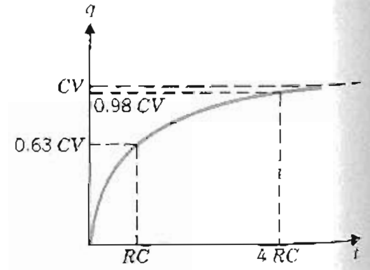


FIGURE 15-20 Charge accumulation on a capacitor in an RC circuit as a function of time.

PROBLEMS

15.1 A wire carries a current $i = 1$ A. How many electrons pass a fixed cross section of the wire in 1 sec?

15.2 Copper has one conduction electron per atom, that is, each atom contributes one electron that is free to move through the solid. The density of copper is 9 g/cm^3 and its molecular weight is 64 g/mole. A wire carries a current of 10 A. The cross-sectional area of the wire is 3 mm^2. (a) What is the current density? (b) What is the number of conduction electrons per m^3? (c) What is the drift velocity?

Answer: (a) 3.33×10^6 A/m^2, (b) 8.47×10^{28} m^{-3}, (c) 2.46×10^{-4} m/sec.

15.3 A copper wire 15 m long has 8×10^{26} mobile electrons. What is the drift velocity of the electrons if the current in the wire is 5 A?

15.4 The resistivity of copper at ambient temperature is $\rho = 1.7 \times 10^{-8}$ Ω-m. What is the resistance of a copper wire 5 m long and 2×10^{-3} m in diameter?

15.5 A copper wire ($\rho = 1.7 \times 10^{-8}$ Ω-m) 10 m long and 1×10^{-3} m in diameter carries a current of 2 A. What is the potential difference across the ends of the wire?

15.6 In the earth's atmosphere positive charges move toward the earth and negative charges move away from it. The total current is approximately 1800 A. The average value of the electric field responsible for this current near the surface of the earth is 100 N/C. What is the resistivity of the air at the surface of the earth? The radius of the earth is 6.37×10^6 m.

Answer: 2.83×10^{13} Ω-m.

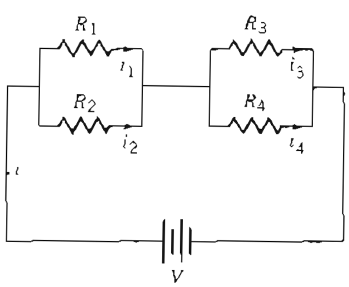


FIGURE 15-21 Problem 15.8.

15.7 In the circuit of Example 15-4, let $R_1 = 5\,\Omega$, $R_2 = 10\,\Omega$, $R_3 = 4\,\Omega$, and $V = 2$ V. Find the currents i_1, i_2, and i_3.

15.8 In the circuit of Fig. 15-21 (a) Find the currents through each resistor. $R_1 = 3\,\Omega$, $R_2 = 6\,\Omega$, $R_3 = 6\,\Omega$, $R_4 = 12\,\Omega$, $V = 18$ V. (b) What is the total current i?

15.9 The current through R_3 in the circuit of Fig. 15-22 is 0.2 A. (a) What is the current in R_1, R_2, and R_4? (b) What is the voltage of the battery?

Answer: (a) 1.1 A, 0.5 A, 0.4 A, (b) 4.2 V.

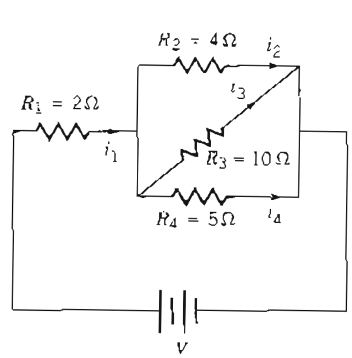


FIGURE 15-22 Problem 15.9.

15.10 How many possible resistance values can be obtained with three resistors $R_1 = 50\,\Omega$, $R_2 = 100\,\Omega$, and $R_3 = 150\,\Omega$?

15.11 In the circuit of Fig. 15-23, find i_B, the voltage drops across R_1 and R_2, and the voltage differences $V_C - V_D$ and $V_D - V_E$.

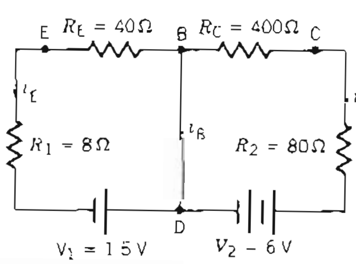


FIGURE 15-23 Problem 15.11

15.12 The current i through R_1 in the circuit diagram of Fig. 15-24 is 40 mA. (a) What is the current through R_2, R_3, and R_4? (b) What is the potential difference between A and B?

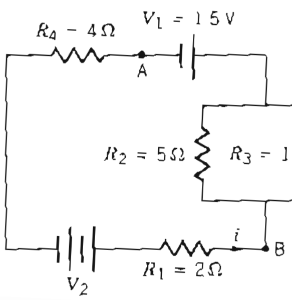


FIGURE 15-24 Problem 15.12.

15.13 The voltage drop across R_3 in the circuit diagram of Fig. 15-25 is 4 V. (a) Find the currents through the resistor R_1, R_2, and R_3. (b) What is the resistance of R_2?

Answer: (a) 0.5 A, 0.3 A, 0.8 A, (b) 1.67 Ω.

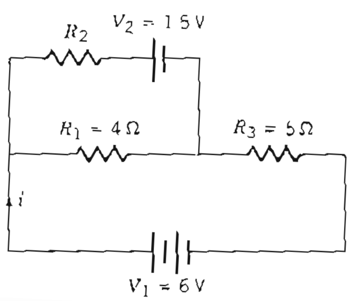
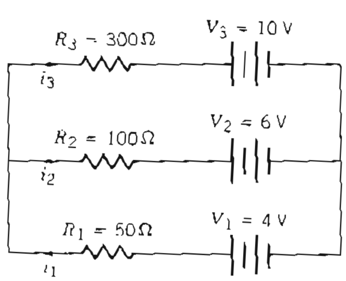


FIGURE 15-25 Problem 15.13.

15.14 Find the currents through the resistors R_1, R_2, and R_3 of the circuit of Fig. 15-26.



15.15 The circuit of Fig. 15-27 is known as the Wheatstone Bridge. It is used to find the resistance of an unknown resistor R_x in terms of three known resistors R_1, R_2, and R_s. The value of R_s is adjusted until no current flows through the galvanometer G. (The arrow over the resistor symbol of R_s indicates that R_s is a variable resistor.) Let $R_1 = 10\,\Omega$ and $R_2 = 100\,\Omega$. If no current flows through G when $R_s = 470\,\Omega$, what is the value of R_x?

Answer: 4700 Ω.

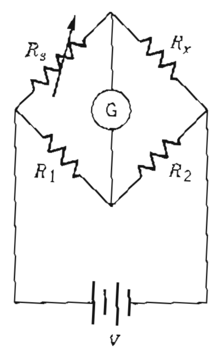


FIGURE 15-27 Problem 15.15.

15.16 A galvanometer has an internal resistance of 2000 Ω and a current of 50 μA will cause full-scale deflection. What shunt resistance is required to use it as an ammeter whose full scale reads 0.1 A?

Answer: 1 Ω.

15.17 The galvanometer of problem 15.16 is to be used as a voltmeter with a maximum scale reading of 10 V. What series resistance is required?

Answer: 1.98×10^5 Ω.

As a rough approximation in the following four problems treat the 120 V ac (alternating current) voltage as a 120 V dc (direct current) constant voltage source, i.e., as a 120 V battery.

15.18 An electric light bulb marked 100 W is used in a home in which the wall outlet is at 120 V. What is the resistance of the filament in the bulb?

15.19 An immersion heater draws 3 A when it is plugged in a 120-V wall outlet. What is the power consumption of the heater?

15.20 If the immersion heater of problem 15.19 is used to boil a cup of water ($m = 150$ g) initially at 27°C and 80% of the power is absorbed by the water, how long will it take for the water to boil?

15.21 An electric heater of resistance 5 Ω is plugged in a 120-V outlet by means of an extension line. Compare the power loss in an extension line 5 m long when the line is made of No. 12-gauge copper wire (2.5 mm in diameter) and when it is made of No. 14-gauge wire (1.6 mm in diameter). The resistivity of copper is 1.7×10^{-8} Ω-m.

Answer: 9.90 W, 23.96 W.

15.22 A megawatt (10^6 watts) of electrical power is needed to run a factory. Compare the energy losses in the transmission lines when the voltage is 120 V with when it is 6000 V.

15.23 A resistor $R = 1000$ Ω and a capacitor $C = 100\ \mu$F are connected in series with a 10-V battery and a switch. (a) How long after closing the switch will the voltage across the capacitor be 1.0 V. (b) When will the capacitor be charged to 99% of its final charge?

Answer: (a) 1.05×10^{-2} sec, (b) 0.46 sec.

15.24 A capacitor C in series with a resistor R is charged by turning the switch to position a in Fig. 15-28. After the capacitor has been charged, the switch is returned to position b. Find an expression for the charge on the capacitor as a function of time. Take $t = 0$ at the moment the switch is changed from a to b.

Answer: $q = CV\,\varepsilon^{-t/RC}$.

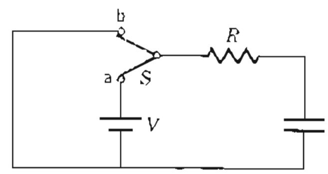


FIGURE 15-28 Problem 15.24.

16.1 INTRODUCTION

Almost everyone has performed elementary experiments with bar magnets. If the bar magnet is suspended by a thread or supported by a pivot, one of the ends will point in a northerly direction. This end of the magnet is called the *north pole* of the magnet, with symbol N. The opposite end of the magnet is called the *south pole*, with symbol S. Elementary experiments also show that like poles repel and unlike poles attract. This suggests that there is something that we call a *magnetic field* by which poles can exert forces on each other. This field is similar to the two other fields we have already considered, the gravitational field and the electric field. There is one important difference, however: If we break a bar magnet in half, we cannot make single poles, but instead we will have two bar magnets. The broken end becomes the south pole of the half that has the north pole, and the other broken end becomes the north pole of the half that has the south pole.

There is an intimate relation between the motion of electric charges and magnetic fields, and our technological society is largely based on this relationship, from the generation of electric power to many types of electronic devices. We will not deal with all these; instead we will consider only those effects which we need for the understanding of the concepts of modern physics presented in later chapters, namely, the magnetic field of a wire coil, the magnetic moment of a current loop, the force of a magnetic field on a moving charge, and the nature of electromagnetic waves.

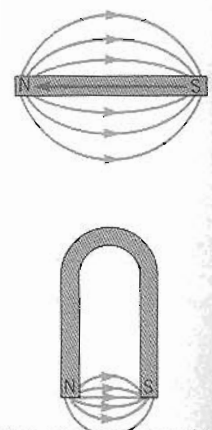


FIGURE 16-1 ■ Magnetic field lines in between the poles of a bar magnet and a horseshoe magnet.

16.2 MAGNETIC FIELDS

We may map magnetic fields by using a small compass that we will represent by a small arrow with its head as the north pole of the compass magnet. We arbitrarily define the direction of the magnetic field at a given point to be the direction in which the compass points. Figure 16-1 shows the fields of a bar magnet and a horseshoe magnet. The fields are indicated by continuous lines from the north to the south pole, and the number of lines is arbitrary, although in the comparison of two magnets the stronger one is customarily represented by a greater number of field lines. The direction of the magnetic field at a given point is the tangent to the field line at that point. In future drawings we may represent the field direction by a single arrow on a line.

Hans Christian Oersted (1777–1851).

In 1820 the Danish physicist Hans Oersted (1777–1851) found that there is a magnetic field associated with current flowing in a wire. The direction of the magnetic field for a long, straight wire is schematically represented in Fig. 16-2. The field lines are circular about the wire. There are many concentric field lines, but the field becomes weaker as we move away from the wire. The direction of the magnetic field is determined by a right-hand rule. If the thumb of the right hand is pointed in the direction of the current and the fingers are curled, the circular direction of the fingers is the direction of the magnetic field.

Suppose we have a circular loop of wire that carries current, as in Fig. 16-3. If we apply the right-hand rule, we see that one side of the loop has a north pole perpendicular to the plane of the loop and its opposite side will become a south pole. The arrows represent the magnetic field direction *inside* the loop. These field lines return *outside* the loop so that the loop itself becomes a magnet. This situation will remain regardless of the shape of the current loop; that is, a rectangular loop will give the same result.

16.3 FORCE ON CURRENT-CARRYING WIRES

In the preceding sections, we have presented some qualitative facts about magnetic fields. We have seen that magnets exert forces on other magnets. We have also seen that a wire carrying a current produces a magnetic field, that is, becomes a magnet.

Experiment shows that when a wire carrying a current is placed in a magnetic field, it will experience a force. We can use this to define the magnitude of a magnetic field, **B**. (Remember that the direction of **B** has been defined as the direction taken by the north pole of a compass).

Figure 16-4 is a schematic drawing of an experiment which shows that when a wire carrying a current is placed in a magnetic field **B** the force **F** is in a direction that is *perpendicular* to the plane defined by the field and the direction of the current. We also find experimentally that there is no force on a wire if the wire is in the direction of the magnetic field and that the force **F** is proportional to the sine of the angle θ between the field and the wire. The experiment also shows that the force on the wire is proportional to the current in the wire i and to the length of wire Δl in the field. From these experimental results we can define the magnitude of the magnetic field **B** as follows:

$$B = \frac{F}{i\,\Delta l \sin\theta} \tag{16.1}$$

It should be clear that Eq. 16.1 defines B unambiguously because, if we double or cut in half the current, the force on the wire will change accordingly. Similar arguments apply to Δl and $\sin\theta$.

From Eq. 16.1, the SI unit for B is newtons/ampere-meter (N/A-m). The name for this unit of B is the tesla (T). An older unit for the magnetic field still in use is the gauss (G), where 1 tesla = 10^4 gauss. The earth's magnetic field is about 10^{-4} T, so that 1 tesla is a large quantity. Having defined the magnetic field by Eq. 16.1, we can now state that when a segment of wire Δl, carrying a current i, is placed in a magnetic field of magnitude B, it will experience a force

$$F = i\,\Delta l\,B \sin\theta \tag{16.2}$$

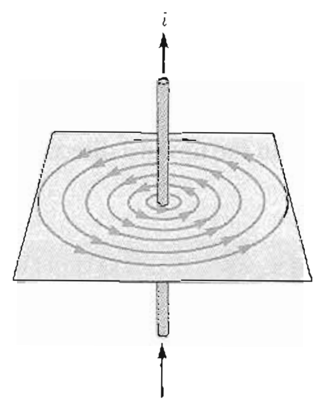


FIGURE 16-2 Magnetic field lines around a long, straight wire carrying a current i.

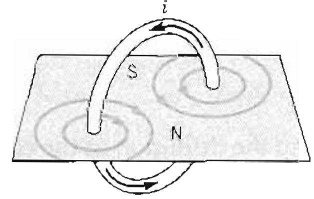


FIGURE 16-3 Magnetic field lines on a plane through a current-carrying circular wire loop.

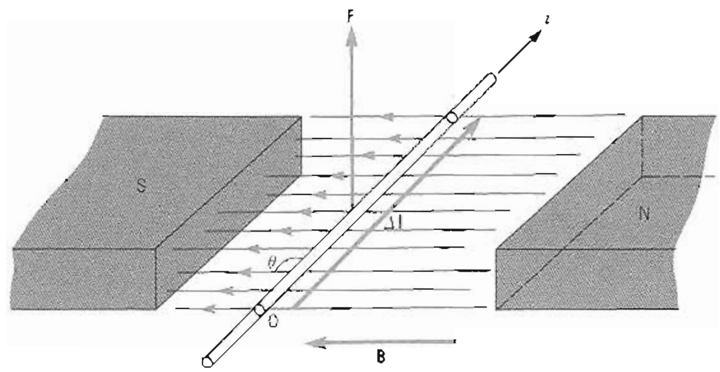


FIGURE 16-4 Force on a current-carrying wire of length Δl in a magnetic field.

The direction of this force is the perpendicular both to the magnetic field and to the direction of the current. We recognize that this relationship can be represented by a vector cross product. That is, Eq. 16.2 may be written as

$$\mathbf{F} = i\,\Delta\mathbf{l} \times \mathbf{B} \tag{16.3}$$

It is conventional to let the current i be a scalar quantity and to let Δl be a vector pointing in the direction of the current. If we let the element of wire in Fig. 16-4 measure 0 at the reader's end and let Δl be the vector length in the magnetic field, then the right-hand rule for vector cross product discussed in Chapter 2 will yield a force in the upward direction perpendicular to the plane of vectors Δl and $\mathbf{B}$.

16.4 TORQUE ON A CURRENT LOOP

We are now able to understand the operation of the galvanometer that was used in Chapter 15 to construct ammeters and voltmeters.

Consider a single rectangular loop of wire connected to a pivot rod, as shown in Fig. 16-5*a*. Let the length of sides 1 and 3 be a and that of sides 2 and 4, b. We can use Eq. 16.3 to find the force on each side of the loop. For sides 1 and 3 the magnitudes of the forces are the same because the angle between Δl and $\mathbf{B}$ is 90° and both wires have the same length a, that is,

$$F_1 = F_3 = i\,\Delta l\,B \sin 90^\circ = iaB \tag{16.4}$$

From the definition of the cross product, we see that in Fig. 16-5*a* $\mathbf{F}_1$ is out of the page toward the reader whereas $\mathbf{F}_3$ is into the page. These two forces are drawn in Fig. 16-5*b*. Similarly the magnitudes of the forces on sides 2 and 4 are equal.

$$F_2 = F_4 = i\,\Delta l\,B \sin(90^\circ - \theta) = ibB \sin(90^\circ - \theta)$$

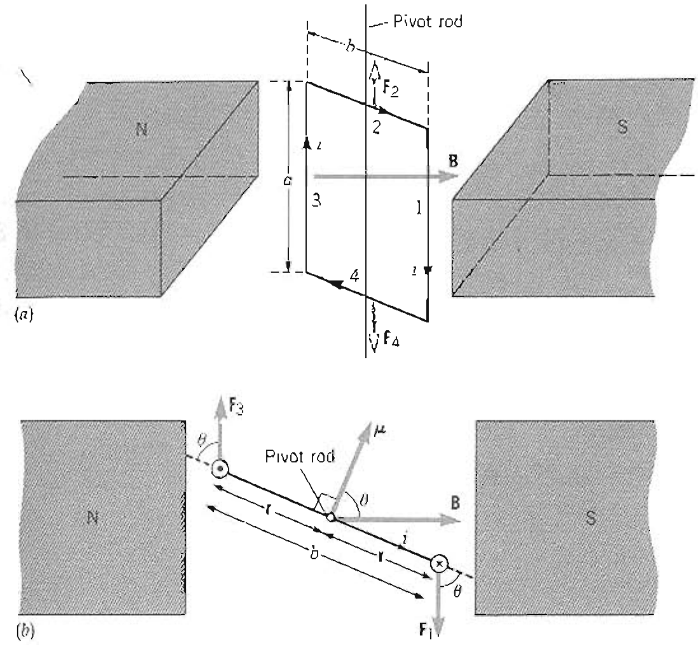


FIGURE 16-5 Torque on a current-carrying rectangular loop of wire on a pivot rod when placed in a magnetic field. (*a*) Side view. (*b*) Top view.

The direction of $\mathbf{F}_2$ in Fig. 16-5*a* is upward, while that of $\mathbf{F}_4$ is downward.

We conclude that there is no net force in any direction. However, if we look at the top view (Fig. 16-5*b*) along the pivot rod we see that there exists a torque that tends to rotate the loop about the pivot rod. The torque is given by Eq. 8.5.

$$\tau = \mathbf{r} \times \mathbf{F} \qquad (8.5)$$

Applied to the present situation, we see that $\mathbf{F}_1$ and $\mathbf{F}_3$ exert a torque on the loop. The net torque is the sum of the individual torques caused by $\mathbf{F}_1$ and $\mathbf{F}_3$, but because they are equal, we simply multiply the torque exerted by one of these forces by two, or

$$\tau = 2\mathbf{r} \times \mathbf{F}$$

$$\tau = 2r\,F \sin\theta \qquad (16.6)$$

where $\mathbf{F}$ stands for either $\mathbf{F}_1$ or $\mathbf{F}_3$. Substituting Eq. 16.4 for F in Eq. 16.6, we obtain

$$\tau = 2r\,iaB \sin\theta \qquad (16.7)$$

From Figs. 16-5*a* and 16-5*b* we see that $r = b/2$; therefore

$$\tau = iabB \sin\theta \qquad (16.8)$$

We recognize the product ab as the area of the loop. Calling this area A, Eq. 16.8 may be written as

$$\tau = iAB \sin\theta \qquad (16.9)$$

Although Eq. 16.9 has been derived for a rectangular loop of wire, it can be shown that the result is the same for any other geometric configuration.

Equation 16.9 tells us that the torque is a maximum when $\theta = 90°$, that is, when the plane of the loop lies in the direction of B, and it is zero when $\theta = 0°$, that is, when B is perpendicular to the plane of the loop. The magnitude of the torque can be increased by increasing the current in the loop. Because in the galvanometer this torque on the coil is opposed by the twisting (torsion) torque of the metal strip used to suspend the coil, the magnitude of the angle of rotation is a function of the current passing through the coil. An alternative way of increasing the torque on the coil is by using a coil made of several loops. To increase the sensitivity of the galvanometer at low currents, a coil of many loops of wire is used.

16.5 MAGNETIC DIPOLE MOMENT

In the preceding section, we saw that the important element in determining the torque on a wire loop, for a given field **B** in which it is placed, is the product of the area of the loop and the current through the loop (see Eq. 16.9). This quantity is called the ***magnetic dipole moment*** or simply the ***magnetic moment*** of the coil with symbol μ, where

$$\mu = iA \qquad (16.10)$$

The expression for the torque can now be written as

$$\tau = \mu B \sin\theta \qquad (16.11)$$

Equations 16.11 and 16.9 give the magnitude of the torque, but they do not specify the direction of τ. The direction of the torque can be obtained by using Eq. 16.5

$$\tau = 2\mathbf{r} \times \mathbf{F} \qquad (16.5)$$

From the definition of the cross product, the direction of τ in Fig. 16-5b is the perpendicular to the paper directed inward. We can specify both the magnitude and the direction of τ in terms of the magnetic moment by assigning a vector direction to μ. We define μ as a vector whose magnitude is given by Eq. 16.10 and whose direction is the perpendicular to the plane of the loop according to the right-hand rule. That is, we curl the fingers of the right hand in the direction of the current and the extended thumb indicates the direction of μ (see Fig. 16-5b). We can now express the torque on the loop as

$$\tau = \mu \times B \tag{16.12}$$

From the definition of the cross product, it is clear that Eq. 16.12 yields the correct value for the magnitude (Eq. 16.11) and for the direction (Eq. 16.5) of τ.

Because a magnetic dipole experiences a torque when placed in an external magnetic field, work must be done by an external agent to change its orientation. As in the cases considered earlier (gravitational and electrical), this work, by definition, becomes the potential energy E_p of the dipole. Recalling that only *changes* in potential energy are experimentally observed, we must define a zero or reference orientation. It is customary to set $E_p = 0$ when $\theta = 90°$, that is, when the dipole vector is perpendicular to the magnetic field. To calculate E_p for any other orientation of μ, we calculate the work using Eq. 8.13

$$W_0 = \int_{\theta_0}^{\theta_f} \tau \, d\theta \tag{8.13}$$

Setting this work equal to E_p, and substituting Eq. 16.11 for τ in Eq. 8.13,

$$E_p = \int_{90°}^{\theta} \mu B \sin\theta \, d\theta$$

$$E_p = -\mu B \cos\theta \Big|_{90°}^{\theta}$$

$$E_p = -\mu B \cos\theta \tag{16.13}$$

This expression for E_p can be written as a dot product (see Eq. 2.1), that is,

$$E_p = -\mu \cdot B \tag{16.14}$$

We should notice that, because the $\cos\theta$ varies between 1 and -1, the maximum energy is μB. This occurs when $\cos\theta = -1$ or $\theta = 180°$, that is, when μ and B are antialigned. When μ and B are aligned, $\theta = 0°$, $\cos\theta = 1$, and the potential energy is at its minimum value of $E_p = -\mu B$.

EXAMPLE 16-1

Assume that the electron in a hydrogen atom is essentially in a circular orbit of radius 0.5×10^{-10} m, and rotates about the nucleus at the rate of 10^{14} times per second. What is the magnetic moment of the hydrogen atom due to the orbital motion of the electron?

Solution

$$\mu = \text{area} \times \text{current}$$

$$= \pi r^2 \, i$$

where i is the current due to a single electron. Because current is defined as the amount of charge passing per unit time, we may view the electron's orbit as a racetrack and ask how many times the electron passes a given point per second. The current is simply

$$i = e\nu$$

where ν is the frequency of rotation—that is, the number of times the electron passes a given point in its orbit per second—and e is the magnitude of the charge of the electron.

$$\begin{aligned}\mu &= \pi r^2 e\nu \\ &= \pi(0.5 \times 10^{-10}\,\text{m})^2(1.6 \times 10^{-19}\,\text{C})(10^{14}\,\text{Hz}) \\ &= 1.26 \times 10^{-25}\,\text{A-m}^2\end{aligned}$$

Therefore, the hydrogen atom is essentially a small bar magnet and will behave as such in a magnetic field.

16.6 FORCE ON A MOVING CHARGE

We may use the development of Section 16.4 concerning the force on a current-carrying wire in a magnetic field to find the force experienced by a single charge. We saw in Eqs. 15.4 and 15.5 that we may write the current of either a *positive* or *negative* charge carrier as

$$i = qNAv$$

where q was the magnitude of a charge, N the number of charge carriers per unit volume, A the cross-sectional area, and v the average drift velocity of the charge carriers. If we substitute this into Eq. 16.3 we obtain

$$\begin{aligned}\mathbf{F} &= i\,\Delta\mathbf{l} \times \mathbf{B} \qquad (16.3) \\ &= qNAv\,\Delta\mathbf{l} \times \mathbf{B}\end{aligned}$$

We see that $A\,\Delta l$ is the volume of the wire segment that is in the magnetic field B. The product of the charge density N (number of charges per unit volume) and the volume of the wire segment gives the total number of charges. Therefore, if $NA\,\Delta l$ is the number of charges experiencing a total force $\mathbf{F}$, the force per charge carrier $\mathbf{F}_q$ is $F/NA\,\Delta l$ and from Eq. 16.3

$$\mathbf{F}_q = \frac{\mathbf{F}}{NA\,\Delta l} = q\,\mathbf{v} \times \mathbf{B} \qquad (16.15)$$

Note that this has been derived for a positive charge, because we implicitly assume that $\mathbf{v}$ is in the same direction as $\Delta\mathbf{l}$ and therefore the current. If the charge carrier is the electron, the direction of the force is reversed. The force is perpendicular to the plane of v and B by definition of the vector cross product. It is therefore perpendicular to the direction of motion given by the vector v. The definition of work is the dot product of force and displacement $dW = \mathbf{F} \cdot ds = F\, ds \cos\theta$. Because the infinitesimal displacement ds has the same direction as the instantaneous velocity v, and because F is perpendicular to v, $\theta = 90°$, $\cos\theta = 0$, and the magnetic field does no work on the charge. The magnetic field therefore does not change the magnitude of the velocity of the charged particle.

16.7 THE HALL EFFECT

Suppose we have a conducting metal strip of width d and thickness t connected in a circuit and placed in a uniform magnetic field B as in Fig. 16-6. Let the direction of the magnetic field be into the paper, indicated by the symbol ⊗, which suggests the tail of an arrow. The electric field $\mathcal{E}_x$ responsible for the current i will be directed to the right. If we assume for the moment that the current is caused by positive charges, their drift velocity v will be in the direction of $\mathcal{E}_x$ as shown in Fig. 16-6.

Let us consider two points D and C on the metal strip such that the line joining the two points is perpendicular to $\mathcal{E}_x$, (see Fig. 16-6). Without the magnetic field, the potential difference between these two points is zero because no work is done in moving a charge from one point to the other. When the magnetic field is turned on, the drifting charges will experience a force given by Eq. 16.15. We label this force $\mathbf{F}_B$ to indicate that this is the force caused by the magnetic field B.

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} = q\,v\,B \tag{16.15}$$

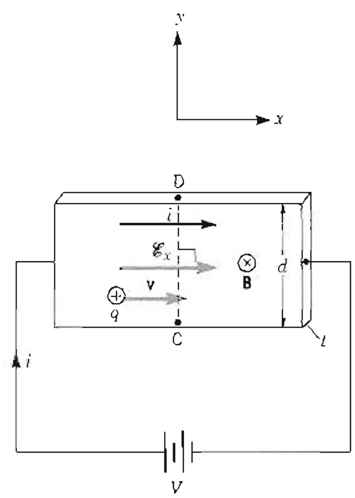


FIGURE 16-6 Experimental arrangement for the measurement of the Hall voltage.

This force, illustrated in Fig. 16-7, causes the positive charges to move to the upper part of the conducting strip while they are moving to the right. Because the sample as a whole must remain neutral, the lower part of the strip will become negatively charged. This situation is also shown in Fig. 16-7. The accumulation of positive charges along the upper part and of negative charges along the lower part creates an electric field $\mathcal{E}_y$ that opposes the further upward drift of positive charges. There will be a potential difference V_H between D and C associated with this electric field. From Eq. 14.19

$$V_H = V_D - V_C = \mathcal{E}_y d \tag{16.16}$$

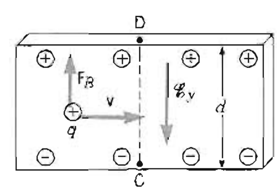


FIGURE 16-7 Behavior of mobile positive charges in the arrangement of Fig. 16-6.

where it is assumed that in equilibrium $\mathcal{E}_y$ is constant and d is the width of the strip (the distance between D and C). This voltage difference is called the *Hall voltage* after the physicist who first measured it, and the phenomenon is called the *Hall effect*. It is clear that the equilibrium Hall voltage V_H will be established when the downward force of $\mathcal{E}_y$ equals the upward force resulting from the magnetic field. Because the force of the electric field is given by definition in Eq. 14.1 as $F_{\mathcal{E}} = q\mathcal{E}_y$, we can say that at equilibrium (which is quickly established)

$$F_{\mathcal{E}} = F_B \tag{16.17}$$

$$q\mathcal{E}_y = q\,v\,B \tag{16.18}$$

therefore

$$\mathcal{E}_y = v\,B$$

Substituting for $\mathcal{E}_y$ in Eq. 16.16, we obtain

$$V_H = v\,B\,d \tag{16.19}$$

Because the Hall voltage can be readily measured by connecting a voltmeter between D and C, the Hall effect permits the experimental determination of the drift velocity of the charge carriers. We can obtain additional information if we use Eqs. 15.4 or 15.5 as follows

$$i = qNAv$$

or

$$v = \frac{i}{qNA} \tag{16.20}$$

Substituting Eq. 16.18 for v in Eq. 16.17, we obtain

$$V_H = \frac{i\,B\,d}{q\,N\,A} \tag{16.21}$$

Note that A is the cross-sectional area of the foil, hence A = thickness $(t) \times$ width (d). Therefore

$$V_H = \frac{1}{qN}\frac{iB}{t} \tag{16.22}$$

or

$$V_H = R_H\frac{iB}{t} \tag{16.23}$$

where $R_H = 1/qN$ is called the *Hall coefficient*. Because i, B, and t are measurable, the magnitude of the Hall voltage will yield the value of N, the density of charge carriers. In the SI system of units, this density will be the number per cubic meter.

Additional important information can be obtained from the Hall effect. In our discussion, we assumed that the charge carriers were positively charged. These charges were deflected toward the upper part of the foil, raising the potential of point D with respect to point C. Suppose, however, that the charge carriers are negatively charged particles. In this case, the drift velocity of the carriers will be opposite to that of $\mathcal{E}_x$, as in Fig. 16-8. Although the velocity vector v is reversed, the direction of the force given by Eq. 16.15 is still upward because the charge q is negative. As a result, the upper part of the strip will have an accumulation of negative charges and the lower part an

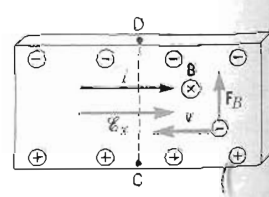


FIGURE 16-8 Behavior of mobile negative charges in the Hall effect experiment.

accumulation of positive charges. Point D will now be at a lower potential than point C. The polarity of the Hall voltage will tell which type of carrier is responsible for conduction. We will see in Chapter 25 that the semiconductors used in logic circuits can be made to have either positive or negative charge carriers.

EXAMPLE 16-2

A current of 50 A is established in a slab of copper 0.5 cm thick and 2 cm wide. The slab is placed in a magnetic field B of 1.5 T. The magnetic field is perpendicular to the plane of the slab and to the current. The free electron concentration in copper is 8.4×10^{28} electrons/m^3. What will be the magnitude of the Hall voltage across the width of the slab?

Solution Using Equation 16.19

$$V_H = \frac{1}{Nq}\frac{iBd}{A}$$

$$= \frac{50\,\text{A} \times 1.5\,\text{T} \times 2 \times 10^{-2}\,\text{m}}{8.4 \times 10^{28}\,\text{m}^{-3} \times 1.6 \times 10^{-19}\,\text{C} \times 10^{-4}\,\text{m}^2}$$

$$= 1.12 \times 10^{-6}\,\text{V}$$

16.8 ELECTROMAGNETIC WAVES: THE NATURE OF LIGHT

In 1670, Christian Huygen proposed that the propagation of light could be explained by assuming that light is a wave. Huygen's theory was not widely accepted until 1801, when Thomas Young performed the first successful experiment that exhibited the interference of light. Even though Young's experiment established firmly the wave nature of light, one important question remained unanswered: What is the nature of the light wave?

Starting with the fundamental laws of electromagnetism, James Maxwell (1831–1879) in 1873 showed that accelerated charges would produce electromagnetic waves whose velocity of propagation c through free space should be

$$c = 3 \times 10^8\,\text{m/sec}$$

James Clerk Maxwell (1831–1879).

Maxwell and other physicists of that period also showed that an electromagnetic wave consists of an electric field $\mathcal{E}$ and a magnetic field B perpendicular to each other with both $\mathcal{E}$ and B perpendicular to the direction of their propagation. The spatial and temporal behavior of the electric and magnetic fields is identical to the traverse motion of the particles in a string when a traveling wave propagates through it.

Suppose an electromagnetic wave travels along the x axis. If we measure the value of $\mathcal{E}$ and B at different points along the x axis, at some fixed time t we will

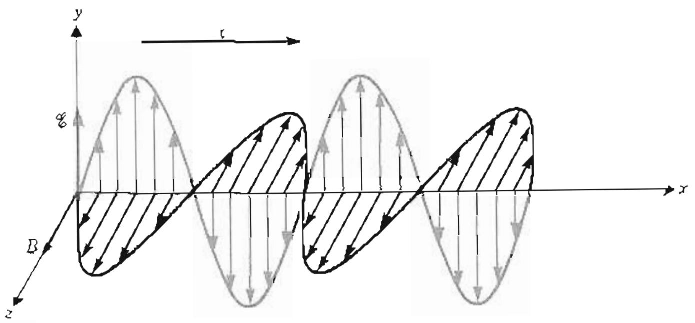


FIGURE 16-9 In an electromagnetic wave, the electric and the magnetic fields are at right angles to each other and to the direction of propagation of the wave.

observe that both $\mathcal{E}$ and B vary sinusoidally with x. This behavior is illustrated in Fig. 16-9. Similarly, if we sit at a fixed point in space and measure $\mathcal{E}$ and B at that point as a function of time, we observe that both vary sinusoidally with time.

This behavior of the electric and magnetic fields can be represented mathematically as

$$\mathcal{E} = \mathcal{E}_0 \sin(kx - \omega t)$$

and (16.22)

$$B = B_0 \sin(kx - \omega t)$$

Equations 16.22 have the same mathematical form as the sinusoidal traveling wave that was introduced in Chapter 11 (see Eq. 11.4 and following). James Maxwell showed that any charge distribution that oscillates sinusoidally with time should produce electric and magnetic fields that behave as described by Eqs. 16.22. Moreover, the frequency ω of the electromagnetic wave should be the same as the frequency of oscillation of the charges producing it. We should indicate at this point that no motion of material particles is involved in the electromagnetic wave, hence, there is no need for a medium of propagation.

One of the key characteristics of a particular type of wave is its velocity of propagation. Maxwell's theoretical prediction was that all electromagnetic waves should travel with velocity $c = 3 \times 10^8$ m/sec. Within the experimental uncertainty this was the value that was measured for the speed of light in 1849. This fact led Maxwell to postulate that light is an electromagnetic wave. Fifteen years after Maxwell's calculations, Heinrich Hertz (1857–1894) was able to produce waves of electromagnetic origin using a circuit with an oscillating current flowing through it. Hertz found that the speed of his electromagnetic waves agreed, within the experimental uncertainties, with the value predicted by Maxwell. The common

Heinrich Hertz (1857–1894).

electromagnetic spectrum is listed in the following table, although there are no upper or lower limits.

Electromagnetic Spectrum

Name	Frequency (Hz)	Wavelength (m)
Gamma rays	10^{23}-10^{19}	10^{-14}-10^{-10}
X rays	10^{20}-10^{16}	10^{-12}-10^{-8}
Ultraviolet rays	10^{17}-10^{15}	10^{-9}-10^{-6}
Visible light	$(4\text{-}7.5) \times 10^{14}$	$(7.5\text{-}4) \times 10^{-7}$
Infrared rays	10^{14}-10^{11}	10^{-5}-10^{-4}
Microwaves	10^{12}-10^{9}	10^{-4}-10^{-1}
Short radio waves	10^{9}-10^{6}	10^{-3}-10^{-2}
FM, TV	10^{8}	1
AM radio	10^{7}-10^{6}	10^{2}-10
Long radio waves	10^{6}-10^{1}	10^{3}-10^{7}

The laws of electromagnetic waves apply to waves of the entire electromagnetic spectrum. Some wavelengths, such as visible light, are more accessible for experiment than are other wavelengths, but all have been at least partially checked for consistency with this model.

PROBLEMS

16.1 What force is experienced by a wire of length $l = 0.08$ m at an angle of 20° to the magnetic field direction carrying a current of 2 A in a magnetic field of 1.4 T?

16.2 The earth's magnetic field at the equator is 4×10^{-5} T and is parallel to the surface of the earth in the south-north direction. (Note that the earth's geographic north pole is the magnetic south pole.) A wire 2 m long of mass $m = 9$ g is suspended by a string. The wire is also parallel to the earth's surface and carries a current of 150 A in the east-west direction. (a) What is the tension on the string? (b) What would be the tension if the current was in the west-east direction?

Answer: (a) 10.02×10^{-2} N, (b) 7.62×10^{-2} N.

16.3 A rectangular wire loop carrying a current $i = 5$ A is hung by a string from the end of a rod pivoted about its midpoint, as in Fig. 16-10. The lower part of the loop is in a region where there is a uniform magnetic field $B = 2$ T perpendicular to the plane of the loop as shown. What weight must be placed on the other end of the pivoted rod to balance it?

Answer: 8 N.

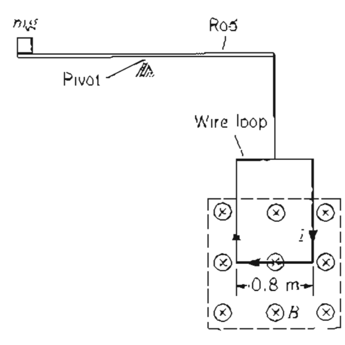


FIGURE 16-10 Problem 16.3.

16.4 The wire of Fig. 16-11 carries a current $i = 2$ A. Find the force on each segment of the wire when it is placed in a region where there is a magnetic field $B = 1.5$ T directed along the positive y axis.

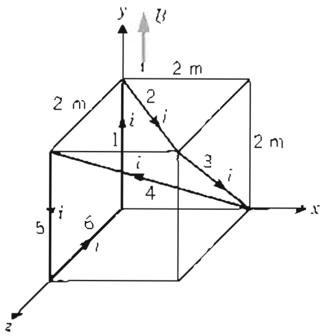


FIGURE 16-11 Problem 16.4.

16.5 The wire loop of Fig. 16-12 carries a current of 2 A. It is placed in a region where there is a magnetic field $B = 0.5$ T parallel to the plane of the loop. (a) Calculate the force on each side of the wire loop. (b) What is the torque on the wire loop?

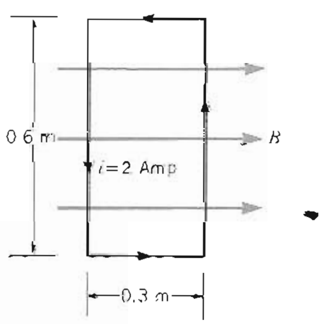


FIGURE 16-12 Problem 16.5

16.6 The wire loop of Fig. 16-13 carries a current $i = 10$ A. It is placed in a uniform magnetic field $B = 1.2$ T. Let $r = 2$ m. (a) Find the net force on the circular part of the loop. (b) What is the net force on the loop?

Answer: (a) 48 N, in the upward direction, (b) 0.

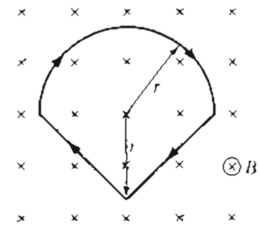


FIGURE 16-13 Problem 16.6.

16.7 What is the maximum torque that acts on a coil of wire that consists of 10 loops of diameter 0.04 m and carries a current of 2×10^{-3} A in a magnetic field of 3×10^{-2} T?

16.8 A wire of length l is to be used to make a coil of one or several circular loops through which a current i will be passed. (a) Show that the maximum magnetic dipole is obtained by making a single loop. (b) What is the dipole moment?

16.9 How much work must be done to rotate the loop of Problem 16.6 from the position shown in Fig. 16-13 to a position where the magnetic field is parallel to the plane of the loop?

16.10 A magnetic dipole μ with a moment of inertia I is placed in a uniform magnetic field B. Initially μ is in the equilibrium position, that is, parallel to B. The dipole is then rotated by a *small* angle θ and then released. (a) Show that the subsequent motion of the dipole is approximately simple harmonic. (b) What is the period of the motion? (*Hint*: For small angles θ, $\sin\theta \approx \theta$.)

16.11 A ring made of an insulator with radius $r = 0.2$ m has a uniformly distributed charge $q = 4 \times 10^{-4}$ C. The ring is placed in the x-y plane of a region where there is a uniform magnetic field $B = 1.2$ T directed along the positive y axis (see Fig. 16-14). The ring is rotated about the z axis with constant angular velocity $\omega = 20$ rev/sec. (a) What is the magnetic moment of the rotating ring? (b) What is the torque exerted by the magnetic field on the ring?

Answer: (a) 1.01×10^{-3} A-m^2, (b) 1.21×10^{-3} N-m.

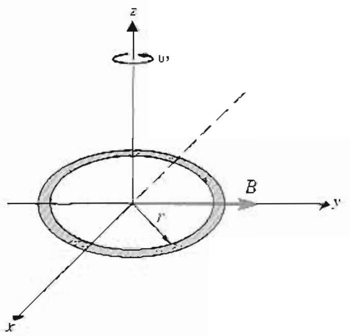


FIGURE 16-14 Problem 16.11.

16.12 A proton is moving with a velocity $\mathbf{v} = (3 \times 10^5\mathbf{i} + 7 \times 10^5\mathbf{k})$ m/sec in a region where there is a magnetic field $\mathbf{B} = 0.4\mathbf{j}$ T. What is the force experienced by the proton?

Answer: $(1.92\mathbf{k} - 4.48\mathbf{j}) \times 10^{-14}$ N.

16.13 A proton is accelerated through a potential difference of 200 V. It then enters a region where there is a magnetic field $B = 0.5$ T. The magnetic field is perpendicular to the direction of motion of the proton. What is the force experienced by the proton?

16.14 A charged particle q is projected in the region between two parallel plates. In the region of the plates there is an electric field $\mathcal{E} = 50,000$ N/C and a magnetic field $B = 0.1$ T. The electric field is perpendicular to the magnetic field, and both are perpendicular to the direction of motion, as shown in Fig. 16-15. If the particle goes through the plates undeflected, what is the velocity of the particle?

Answer: 5×10^5 m/sec.

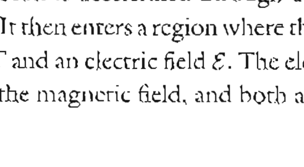


FIGURE 16-15 Problem 16.14.

16.15 A proton is accelerated through a potential difference of 300 V. It then enters a region where there is a magnetic field $B = 0.8$ T and an electric field $\mathcal{E}$. The electric field is perpendicular to the magnetic field, and both are perpendicular to the direction of motion of the proton (see Fig. 16-16). The proton moves through undeflected. What is the value of $\mathcal{E}$?

Answer: 1.92×10^5 N/C.

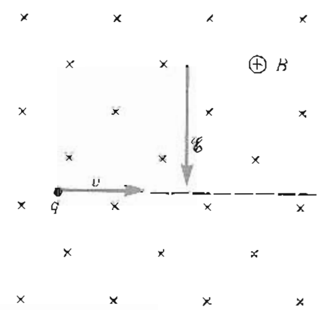


FIGURE 16-16 Problem 16.15.

16.16 A particle of mass $m = 15$ g and charge $q = 3 \times 10^{-3}$ C is moving horizontally near the surface of the earth with a velocity $v = 50$ m/sec. What is the magnitude and direction of the smallest magnetic field B that will keep the particle moving in a straight line? Ignore the electric field mentioned in Problem 15.6.

16.17 An electron is moving with a velocity $\mathbf{v}_1 = (2 \times 10^6\mathbf{i} + 4 \times 10^6\mathbf{j})$ m/sec in a region where there is a uniform magnetic field, it experiences a force $\mathbf{F}_1$ along the z axis. A second electron with velocity $\mathbf{v}_2 = 3 \times 10^6\mathbf{k}$ m/sec experiences a force $\mathbf{F}_2 = 7 \times 10^{-13}\mathbf{i}$ N. (a) What is the direction and the magnitude of the magnetic field? (b) What is the magnitude and direction of $\mathbf{F}_1$?

Answer: (a) $1.46\mathbf{j}$ T, (b) $4.67 \times 10^{-13}\,\mathbf{k}$ N.

16.18 A strip of copper 1 cm wide and 1 mm thick has 50 A of current passing through it. The strip is in a magnetic field of 0.5 T directed into the paper (see Fig. 16-17). The voltage difference $V_H = V_C - V_D = 2 \times 10^{-6}$ V and the observation that V_C is larger that V_D indicates that the conduction is by electrons. What is the density of the electrons responsible for the current? ($e = 1.6 \times 10^{-19}$ C.)

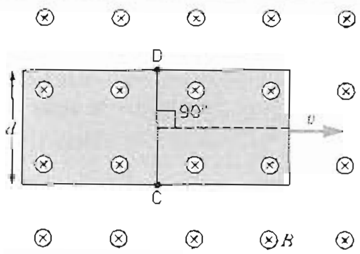


FIGURE 16-17 Problem 16.18.

16.19 A rectangular slab of silicon of thickness 1 mm is used to measure an unknown magnetic field B. The free electron concentration of that particular type of silicon is 6×10^{24} electrons per m^3. When the slab is placed in the region of the magnetic field, perpendicular to the field, and the current in the slab is 20 mA, the Hall voltage is 150 μV. What is the strength of the magnetic field?

Answer: 7.2 T.

16.20 A long metal plate of width $d = 1$ cm is moved with constant velocity v in a region where there is a magnetic field $B = 0.9$ T (see Fig. 16-18). A potential difference $V = 4.5 \times 10^{-3}$ V appears across two points D and C. The line joining the two points is perpendicular to the direction of motion of the plate. What is the velocity v?

Answer: 0.5 m/s.

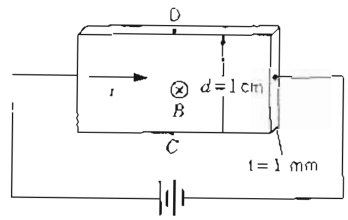


FIGURE 16-18 Problem 16.20.

17.1 INTRODUCTION

By the end of the nineteenth century, most physicists felt rather good about the state of their art. In fact, some felt that their successors would spend their time simply taking measurements to the next decimal place. There were reasons for this complacent attitude. Most of the astronomical data about the motion of the planets, as well as the behavior of ordinary mechanical systems, could be explained using Newton's laws of motion and his law of universal gravitation. The empirical laws concerning electric and magnetic fields had been discovered and fused together by Maxwell, and his prediction of the existence of electromagnetic waves had been experimentally verified by Heinrich Hertz: The nature of light was no longer a mystery. More important, the same laws used to explain the behavior of macroscopic systems were also able to explain the behavior of submicroscopic objects (atoms and molecules). This came about with the development of the techniques of statistical mechanics. By applying Newton's laws statistically the ideal gas law, $PV = nRT$ could be derived. Similarly, the specific heat of gases could be predicted in agreement with the available experimental data.

There were a few minor problems. We will mention two of them that were instrumental in the advent of the scientific revolution that today we call modern physics. The principle of relativity seemed to fail when applied to electromagnetism. The principle states that the laws of physics should be the same in all inertial frames of reference. Someone performing experiments in a spaceship moving with constant velocity with respect to the earth obtains the same results from the experiments as does an experimenter on earth. Because physical laws reflect the results of the experiments, it follows that these laws have to be the same (must have the same mathematical form) in all inertial frames of reference. This mathematical invariance was shown to be preserved with the laws of mechanics, but it broke down with the laws of electricity and magnetism. This "minor" problem eventually led to development of Einstein's *special theory of relativity*.

Another problem that baffled physicists at the beginning of the twentieth century was the nature of the spectrum emitted by a class of objects called *blackbodies*. The predictions of classical ideas did not fit the experimental results. This problem led to the development of what we now call *quantum mechanics*. Relativistic effects do not generally affect computer operation, so we will address only the quantum mechanical part of modern physics in the remainder of this book.

17.2 BLACKBODY RADIATION

All substances at finite temperatures radiate electromagnetic waves. Isolated atoms (in a gas) emit discrete frequencies, molecules emit bands of frequencies, and solids radiate a continuous spectrum of frequencies.

The details of the spectrum emitted by a solid depend on its temperature and to some extent on its composition. At room temperature the spectrum is centered around the infrared; that is, most of the radiation emitted lies in the infrared part of the electromagnetic spectrum. As the temperature of the solid increases, more and more of the emitted radiation is in the visible part of the spectrum; we see it first glow red and then approach white as the temperature is increased.

Objects that emit a spectrum of *universal* character, one that does not depend on the composition of the object, are called blackbodies. The reason for the name is that these objects absorb all the radiation incident on them. They do not reflect light, and hence they appear black. We see them by contrast with other objects or their background. Any object painted with a dull black pigment is a good approximation to an ideal blackbody. Another type of blackbody is a metallic cavity with a small hole (see Fig. 17-1). Any radiation entering the hole has a very small probability of being reflected out, hence the object (hole) is "black." After multiple partial reflections by the inner walls of the cavity, the radiation is eventually absorbed by the atoms in the walls of the cavity. These atoms, in turn, will reradiate electromagnetic waves into the cavity and some of it will leak out through the hole. Theoretically, the character of this radiation that leaks out is the same as that of the other type of blackbody.

17.2a. Character of the Spectrum of a Blackbody

The main features of the spectrum emitted by a blackbody are:

1 The spectrum is continuous with a broad maximum. This fact is shown in Fig. 17-2, which is a plot of $I(\nu)$, the spectral radiancy at each frequency, versus the frequency of the radiation. The spectral radiancy is the energy per frequency emitted per unit time per unit area of the blackbody. The two curves correspond to two different temperatures of the object.

2 The integral of $I(\nu)$ over all ν, which we call I_T, represents the energy emitted per unit time per unit area, regardless of the frequency, and it is found to increase with the fourth power of the temperature. This empirical fact, known as the Stefan-Boltzmann law, states that,

$$I_T = \int_0^\infty I(\nu)\, d\nu = \sigma T^4$$

where the constant $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2\text{-}K^4}$. This integral is clearly the area under the curve for each temperature.

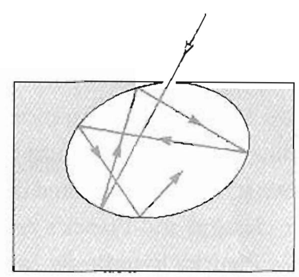


FIGURE 17-1 A metallic cavity with a small hole is an example of a blackbody. Radiation entering the hole is eventually absorbed after successive reflections at the inner walls of the cavity. Some of the radiation reemitted by the atoms in the walls of the cavity leaks out through the hole. This radiation has the same character as that of any other blackbody.

3 Figure 17-2 also shows that the spectrum shifts toward higher frequencies as the temperature increases. In fact, one finds experimentally that the frequency ν_{max}, at which $I(\nu)$ is a maximum, increases linearly with the temperature of the cavity (blackbody), that is,

$$\nu_{max} \propto T$$

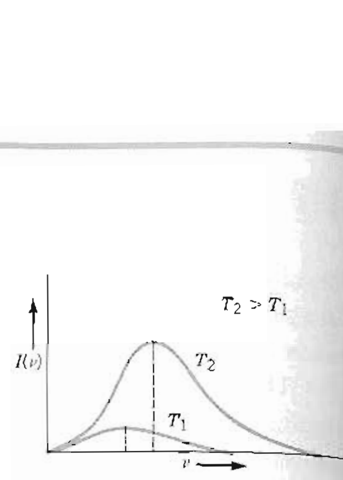


FIGURE 17-2 The intensity of the radiation emitted by a blackbody as a function of the frequency of the radiation for two different temperatures of the blackbody. Note that the total energy (area under the curve) and the frequency at which the intensity is a maximum increase with increasing temperature.

17.2b. Planck's Theory

Attempts by physicists to explain the blackbody spectrum using the laws of classical electromagnetism and thermodynamics proved unsuccessful.

On December 14, 1900, at a meeting of the German Physical Society, Max Planck (1858–1947) presented a paper entitled, "On the Theory of the Energy Distribution Law of the Normal Spectrum." This event is considered the birthday of quantum mechanics. As we will see, these ideas were at first introduced a little bit haphazardly, with no justification other than that they accounted for the experimental facts. Eventually, these ideas were fused together into a set of fundamental principles by Erwin Schrödinger and Werner Heisenberg.

Planck's approach to the problem was to find an empirical mathematical expression for $I(\nu)$ that would fit the experimental data. He then observed that he could derive the expression by making a revolutionary physical hypothesis, namely; ***a system undergoing simple harmonic motion with frequency ν can only have and therefore can only emit energies given by $E = nh\nu$, where $n = 1, 2, 3, \ldots$*** and h is a constant now known as Planck's constant. The value of h, which resulted in a good fit between the data and the expression found by Planck for $I(\nu)$, is 6.63×10^{-34} Joule second (J-sec).

We know that the energy of a harmonic oscillator is proportional to the square of the amplitude of the motion (Section 10.6). In a classical treatment, such as an oscillating spring, this amplitude may vary continuously from zero to infinity. In contrast, Planck postulated that atomic oscillators can have only *discrete* energy values. The classical and Planck's energy spectra for an oscillator are contrasted in Fig. 17-3. By Planck's hypothesis, because an oscillator (such as the atoms in the walls of the cavity) can take only certain values for the energy, when they lose that energy (by emitting electromagnetic waves), they lose it in multiples of $h\nu$. These small quantities of energy are called ***quanta*** (singular, ***quantum***).

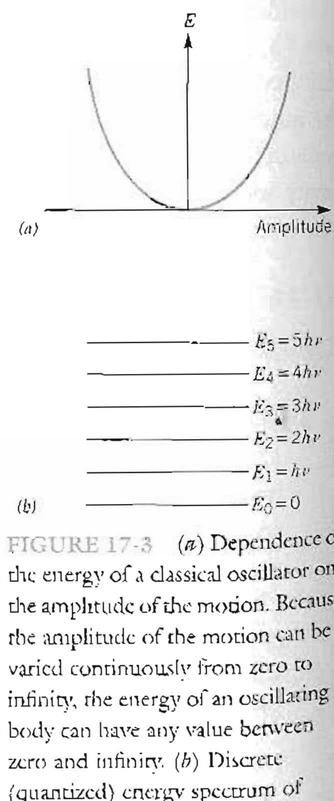


FIGURE 17-3 (*a*) Dependence of the energy of a classical oscillator on the amplitude of the motion. Because the amplitude of the motion can be varied continuously from zero to infinity, the energy of an oscillating body can have any value between zero and infinity. (*b*) Discrete (quantized) energy spectrum of atomic oscillators as proposed by Planck to explain the spectrum of frequencies emitted by a blackbody.

Using the energy spectrum just described for the atomic oscillators, and consequently for the electromagnetic waves emitted by them, together with simple thermodynamic arguments, Planck derived a rather complicated expression for $I(\nu)$ that matched the experimental data:

$$I(\nu) = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T) - 1} \qquad (17.1)$$

where c is the velocity of light, k_B is the Boltzmann constant, ν is the frequency of the electromagnetic wave, and T is the absolute temperature of the blackbody.

Every day experience shows that an oscillator, for example, a pendulum or a mass connected to a spring, stops oscillating progressively and smoothly, not in jumps. Is Planck's hypothesis in conflict with this macroscopic experimental observation? Not really. Let us consider a mass $m = 10$ kg, attached to a spring of force constant $k \doteq 10^3$ N/m. Let the initial amplitude of the motion be $A = 0.1$ m. From elementary mechanics, we know that

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \times 10^3\,\text{N/m} \times (0.1\,\text{m})^2 = 5\,\text{J}$$

and

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 1.59\,\text{Hz}$$

The separation between adjacent energy levels is, by Planck's hypothesis

$$\Delta E = h\nu = 6.63 \times 10^{-34}\,\text{J-sec} \times 1.59\,\text{sec}^{-1} \approx 10^{-33}\,\text{J}$$

This example shows that, for a macroscopic oscillator, the separation between the allowed energy states (as postulated by Planck) is extremely small compared with the energy of the oscillator. The oscillator may be losing energy in jumps, but the effect is not noticeable.

Max Planck (1858–1947).

17.3 THE PHOTOELECTRIC EFFECT

The quantum idea, introduced by Planck to explain the spectrum of a blackbody, was further expanded by Albert Einstein (1879–1955) in connection with the photoelectric effect. Under certain conditions, which we will discuss shortly, light incident on a metal will cause electrons to be ejected from the surface of the metal. This is known as the *photoelectric effect*.

We will summarize an experiment that can be used to study the properties of the electrons ejected from the metal. We will then see the failure of classical ideas to explain the results and, finally, we will introduce Einstein's hypothesis about the nature of electromagnetic radiation and how this hypothesis accounts for the experimental facts.

17.3a. Experimental Facts

An experimental arrangement that can be used to study some of the properties of the photoelectric effect is shown in Fig. 17-4. The apparatus consists of an evacuated tube with two metal plates, C, the cathode, and A, the anode. Monochromatic light (single wavelength) is sent through a quartz window onto the cathode C. Because the anode is at a negative potential V with respect to the cathode, the electrons,

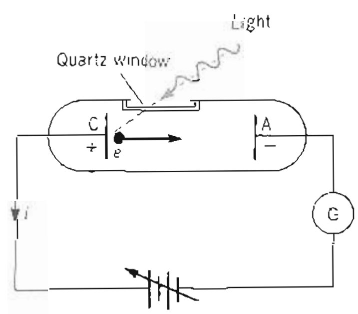


FIGURE 17-4 Experimental arrangement for the study of the photoelectric effect.

on being emitted by the incident light striking the cathode, face a *retarding* voltage V. To reach the anode the photoelectrons must be ejected with a kinetic energy E_k that is greater than the difference in potential energy, $|e|V$, between the anode and the cathode. When $E_k \geq |e|V$, the electrons, on reaching the anode, will be able to contribute to the current through the circuit, which is measured by a galvanometer G. The tube is evacuated to minimize the collisions between the photoelectrons and the gas molecules in the tube. By varying the retarding voltage V, the spectrum of energies with which the electrons are emitted can be determined. Other parameters can be varied to see how they affect the energy and the number of photoelectrons emitted; these include the intensity I and the frequency ν of the incident light and the nature of the cathode. A summary of the experimental results is shown in Fig. 17-5.

If the value of the retarding voltage V and of the frequency of the light are kept constant (Fig. 17-5a), it is seen that the photocurrent i through the circuit increases linearly with increasing intensity I of the incident radiation. This in turn means that the number of electrons emitted with energies $E_k \geq |e|V$ increases with I, because i is proportional to the number of electrons that are collected by the anode.

Figure 17-5*b* shows the dependence of i, and hence of the number of emitted electrons capable of reaching the anode, on the value of the retarding voltage. The experiment is performed while keeping both the intensity and the frequency of the radiation constant. The two curves correspond to two different values of I. The result can be easily understood. For small V's, only the electrons emitted with small energies are turned back by the retarding voltage, and therefore do not contribute to the current i. As V is increased, electrons with higher energies will be turned back, and the current will decrease. When $V = V_0$ (V_0 is called the *stopping potential*), all the electrons, including the most energetic ones, are turned back and the current drops to zero. V_0 is therefore a measure of the maximum energy with which the electrons are ejected from the cathode,

$$|e|V_0 = E_{k\,\max} \tag{17.2}$$

Figure 17-5*b* shows that V_0, and therefore *the maximum energy of the photoelectrons, is independent of the intensity of the light*.

The dependence of V_0 on the frequency ν of the light can be examined by repeating the previous experiment with different ν's. Figure 17-5*c* shows that V_0 (hence $E_{k\,\max}$) increases linearly with ν. That is,

$$V_0 = a\,\nu \tag{17.3}$$

The value of the slope a is found to be 4.1×10^{-15} J-sec/C. The linear dependence of V_0 on ν and the value of the slope remain unchanged if the experiment is repeated with a cathode made of a different metal. Figure 17-5c also shows that for frequencies $\nu \leq \nu_c$ V_0 is zero. This means that no voltage is necessary to stop the most energetic

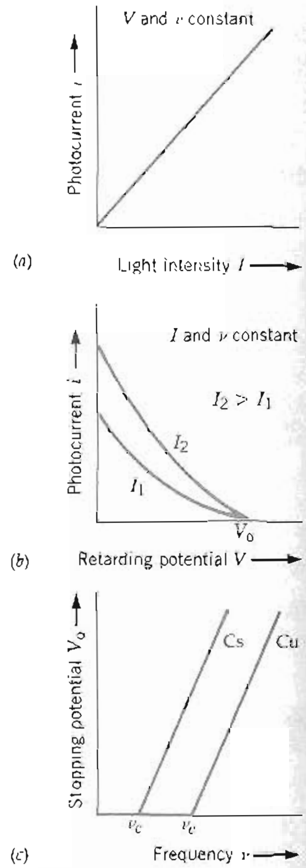


FIGURE 17-5 (*a*) The photocurrent (the number of emitted electrons) increases with increasing light intensity. (*b*) The number of emitted electrons able to reach the anode A decreases as the retarding voltage increases. The stopping potential V_0 is independent of the light intensity. (*c*) The stopping potential increases linearly with increasing frequency of the light. For frequencies below ν_c, V_0 is zero because no electrons are emitted. The value of ν_c depends on the material being illuminated (the cathode). The two curves are for cesium (Cs) and copper (Cu) cathodes respectively.

electrons; no voltage is needed because no electrons are emitted when $\nu \leq \nu_c$. As the graph indicates, the value of ν_c depends on the material used for the cathode.

There is one final experimental fact that is crucial to the discussion that will follow. When the conditions for photoemission are favorable (high enough ν, low enough V), the emission is almost instantaneous. The photocurrent has been observed to occur within 10^{-9} sec from the onset of illumination, which is the limit of experimental accuracy. This essentially instantaneous emission has been observed to take place with extremely low intensities of light, as low as $10^{-10}\ \text{W/m}^2$.

17.3b. Failure of Classical Physics to Explain the Results

According to classical physics, light is an electromagnetic wave (see Section 16.8). To understand the failure of classical physics to explain the experimental results just presented, we need to remind ourselves of two facts about waves:

> The energy of a wave is continuously distributed over the entire space traversed by the wave. For example, when the ripples in the pond move outward from their source, all the water in their path is displaced. The intensity of a wave, which represents the energy carried by the wave per unit area perpendicular to the direction of propagation of the wave, per unit time is proportional to the square of the amplitude of the wave (Eq. 11.20). In the case of electromagnetic waves it can be shown that
>
> $$I = \frac{1}{2}\epsilon_0 c \mathcal{E}_0^2$$
>
> where ϵ_0 is the permittivity of free space, c is the velocity of light, and $\mathcal{E}_0$ is the amplitude of the electric field of the wave.

With these two facts in mind, let us consider an electron that is bound with some energy E_b to the metal. An electric field $\mathcal{E} = \mathcal{E}_0 \sin(kx - \omega t)$ impinges on the bound electron. The electric field will exert a force $F = |e|\mathcal{E} = |e|\mathcal{E}_0 \sin(kx - \omega t)$ on the electrons. This force will do work on the electrons, the amount of work depending on the magnitude of the force. As a result, the electric field will increase the energy of the electrons and, if the energy that the electrons pick up from the electromagnetic field is greater than the binding energy that keeps them in the metal, the electrons will come out of the metal with a kinetic energy E_k, which is the difference between the energy absorbed from the wave and the binding energy E_b. As the amplitude of the wave increases, the magnitude of the force increases, and so does the work done by the electromagnetic field on a given electron. We therefore expect, from classical physics, that the energy given to the electron will increase as the intensity of the wave increases; detailed calculations show that the energy absorbed is, indeed, proportional to the intensity.

Let us now reexamine the experimental results. The results of Fig. 17-5*a* can be explained in terms of classical concepts. The electrons in the metal are bound differently: some more tightly than others. Given a certain intensity of the wave and therefore a certain amount of energy available to them, the electrons will use it to liberate themselves from the metal; any remaining energy will be in the form of kinetic energy of the electrons. For small intensities only those electrons that are weakly bound will come out with sufficient kinetic energy to overcome the retarding potential and to contribute to the current. As the intensity is increased the energy available will increase and more electrons will come out with sufficient energy to reach the anode. The current should increase with increasing intensity, and it does.

The fact that $E_{k\,\max}$ is independent of the intensity is difficult to explain by classical theory. If you increase I, you increase the energy available to all the electrons, including those that are the least tightly bound and that therefore come out with the maximum kinetic energy. Thus the fact that V_0 is independent of I (see Fig. 17-5*b*) cannot be explained by classical ideas.

The fact that $E_{k\,\max}$ increases with ν (see Fig. 17-5c) cannot be accounted for by classical physics. As we have seen, the energy of the electromagnetic wave depends on its intensity (amplitude squared), not on its frequency. Why should V_0 depend on ν? Why is there a ν_c below which no electrons are emitted, no matter how intense the wave is? Classical physics provides no answer.

Finally, the fact that the emission is almost instantaneous plays a key role in the rejection of the classical ideas about the nature of electromagnetic radiation. If we consider radiation with intensity $I = 10^{-10}$ W/m^2, there is no way that the electrons can be emitted in 10^{-9} sec. It should take considerably longer. Let us consider a sheet of some metal with an area of 1 m^2, as shown in Fig. 17-6, and let us assume that light of intensity $I = 10^{-10}$ W/m^2 shines on it. As we mentioned, the energy of the beam is spread continuously over the entire wave front. Let us be optimistic and assume that all the energy falling on a certain atomic site of the metal sheet is absorbed by only one of the electrons of the atom, the most loosely bound. It is well known, from X-ray studies, that the interatomic separation d in a metal is about 2Å ($1\text{Å} = 10^{-10}$ m). We can use this fact to find out how many atomic sites there are in the first layer of the metal surface.

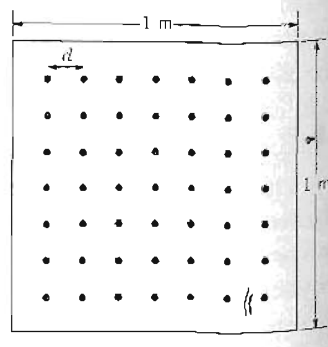


FIGURE 17-6

$$\text{The number of atoms in a 1-m-long row} = \frac{1\text{ m}}{d}$$

$$= \frac{1\text{ m}}{2 \times 10^{-10}\text{ m}} = 5 \times 10^{9}\text{ atoms/row}$$

For simplicity, let the metal have cubic structure. This means that there are as many rows as we have atoms in one row. And, consequently, the number of atoms in the first layer of the metal sheet will be $(5 \times 10^9)^2 = 2.5 \times 10^{19}$ atoms/layer. According to our generous assumption 10^{-10} J/sec-m^2 are shared by 2.5×10^{19} electrons. That is,

$$\text{Energy/second-electron} = \frac{10^{-10}\ \text{J/sec}}{2.5 \times 10^{19}\,e} = 4 \times 10^{-30}\ \text{J/sec-}e$$

It is known from other types of experiments that the minimum binding energy of an electron in a metal is typically a few eV's. Let us take 1 eV. The time required for the electron to collect 1 eV from the electromagnetic wave will be,

$$t = \frac{1\ \text{eV} \times 1.6 \times 10^{-19}\ J/eV}{4 \times 10^{-30}\ \text{J/sec}} = 4 \times 10^{10}\ \text{sec}$$

$$\text{or} \sim 10^5\ \text{days}$$

Thus, we see that classical physics also fails to explain the short time release of electrons in the photoelectric effect.

17.3c. Einstein's Theory

In 1905, five years after Planck's historic paper, Einstein was able to explain the photoelectric effect by proposing a theory about the nature of electromagnetic radiation that was dramatically different from that of classical electromagnetism.

According to Einstein, *the energy of an electromagnetic wave of frequency ν is not continuously distributed over the entire wave front, but instead it is localized in small bundles (particle-like entities) called photons*. The energy of each photon is $E_{\text{photon}} = h\nu$, where h is Planck's constant (see Section 17.2b).

Basically, according to Einstein, a beam of electromagnetic radiation carries energy like a beam of particles, not like a wave. Within the Einstein hypothesis, the intensity of the beam is a measure of the density of photons in the beam. Increasing the intensity without changing the frequency does not change the energy of the individual photons, but rather the number of photons per unit volume of the beam, and thus the energy density of the beam.

Einstein visualized the photoelectric effect as a particle-particle collision in which a photon of energy $h\nu$ collides with an electron in the metal and imparts all its energy to the electron. From conservation of energy,

Albert Einstein (1879—1955).

$$h\nu = E_k + E_b \tag{17.4}$$

where E_b is the energy with which the particular electron is bound to the metal and E_k is the kinetic energy with which that electron is ejected. From Eq. 17.4 it is clear that the value of E_k will depend on how tightly bound a given electron is. The smaller E_b, the larger E_k will be. We can rewrite Eq. 17.4 as

$$E_{k\ \max} = h\nu - \phi \tag{17.5}$$

where ϕ is the minimum binding energy and is called the *work function* of the metal. (Note that ϕ is not an angle but a symbol for energy in this case.) Experimentally,

$E_{k\,max}$ is measured by determining the stopping potential V_0 and $E_{k\,max} = eV_0$, thus Eq. 17.5 can be rewritten as

$$V_0 = \frac{h}{e}\nu - \frac{\phi}{e} \tag{17.6}$$

We can now explain all the experimental data presented at the beginning of this section. The greater the intensity I, the larger the number of photons that will strike the metal cathode every second. This will result in a greater number of photon-electron collisions and a subsequent increase in the number of electrons emitted. Thus the results of Fig. 17-5*a* are explained.

Increasing the intensity increases the number of photons, not the energy of the individual photons. The maximum energy (and therefore V_0) should not depend on the number of photons (on intensity I) but rather on the energy of each photon, that is, on the frequency of the wave. In fact, Eq. 17.6 shows that V_0 should increase linearly with ν. This is in agreement with the result presented in Fig. 17-5*c*. The slope of the curve according to Eq. 17.6 should be equal to $h/e = 4.1 \times 10^{-15}$ J-sec/C, which is the observed value. It should be clear that if the energy of the photons is less than the work function of the metal, no electrons can be ejected. That is, if

$$h\nu < \phi \quad \text{or} \quad \nu < \frac{\phi}{h} = \nu_c \tag{17.7}$$

no photoemission will take place. The cut-off frequency ν_c is accounted for.

Finally, the emission is instantaneous because the process is not one in which the electrons progressively gather energy until they have enough to come out. It is a particle-particle collision. If just one photon with energy $h\nu \geq \phi$ collides with an electron, the latter will be immediately ejected.

EXAMPLE 17-1

The eye is capable of detecting 10 eV of light energy. If we take as the average wavelength of light 6000 Å, how many photons is the eye capable of detecting?

Solution According to Einstein's theory, the energy of a photon is given by $h\nu$. Using this, together with the fact (Eq. 11.2) that the product of the wavelength and the frequency equals the velocity of propagation of the wave, which in the case of light is c, that is,

$$\lambda\nu = c \tag{17.8}$$

we obtain

$$\text{Energy/photon} = h\nu = \frac{hc}{\lambda}$$

$$= \frac{6.63 \times 10^{-34}\ \text{J-sec} \times 3 \times 10^{8}\ \text{m/sec}}{6000 \times 10^{-10}\ \text{m}}$$

$$= 3.32 \times 10^{-19}\,\text{J} = 2.07\,\text{eV}$$

$$\text{Number of photons} = \frac{10\,\text{eV}}{2.07\,\text{eV/photon}} = 5\text{ photons}$$

EXAMPLE 17-2

The cut-off frequency for photoemission in copper is 1.0×10^{15} Hz. What is the maximum kinetic energy of the photoelectrons emitted when light of wavelength 1000 Å is shone on a copper surface?

Solution The work function is given by Eq. 17.7

$$\phi = h\nu_c = 6.63 \times 10^{-34}\,\text{J-sec} \times 1.0 \times 10^{15}\,\text{Hz} = 6.63 \times 10^{-19}\,\text{J}$$

From Eq. 17.5

$$E_{k\,\max} = h\nu - \phi = \frac{hc}{\lambda} - \phi$$

$$= \frac{6.63 \times 10^{-34}\,\text{J-sec} \times 3 \times 10^{8}\,m/sec}{1000 \times 10^{-10}\,\text{m}} - 6.63 \times 10^{-19}\,\text{J}$$

$$= 13.26 \times 10^{-19}\,\text{J} = 8.29\,\text{eV}$$

17.4 FURTHER EVIDENCE FOR THE PHOTON THEORY

There exists today a large number of experimental results that confirm the particle nature of electromagnetic radiation. In this section, we will discuss qualitatively two effects that contribute further evidence to the theory.

17.4a. X-ray Production

In 1895, Wilhelm K. Roentgen (1845–1923) discovered that when highly energetic electrons struck a solid target, a strange (hence the name X rays) kind of radiation

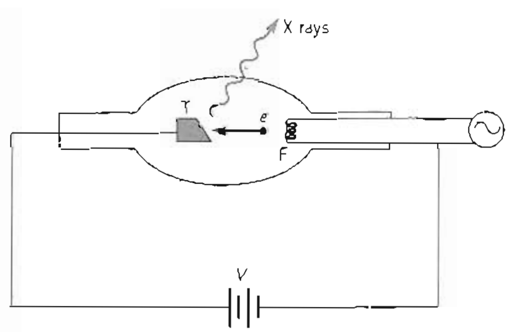


FIGURE 17-7 Apparatus for the production of X rays.

was produced. The radiation was highly penetrating: It passed through objects that were opaque to light. Moreover, the radiation was not deflected by either electric or magnetic fields, indicating that it did not consist of charged particles. The mysterious nature of X rays vanished a few years later when in 1912 Max von Laue (1879–1960) found that they could be diffracted. The diffraction of X rays by crystalline solids was discussed in Chapter 12. These experiments proved that X rays are a form of electromagnetic radiation. Their wavelength, however, is much smaller than that of light waves: typically $\lambda \sim 1$ Å.

Figure 17-7 shows a schematic of the experimental arrangement used to produce X rays. Electrons from a heated filament F are accelerated by a large potential difference V (several thousand volts). As a result, they enter the target T with a kinetic energy $E_k = eV$. On striking the target, X rays are emitted.

An analysis of the spectrum of the emitted X rays reveals several interesting features. We will concern ourselves primarily with one of them that is pertinent to the photon hypothesis. The emission spectrum is continuous, with one very important feature: a sharp, well-defined cut-off on the small wavelength side. These facts are shown in Fig. 17-8, which is a schematic plot of the intensity I of the emitted X rays versus wavelength λ. The value of the cut-off wavelength λ_{min} is independent of the target material but depends on the accelerating voltage V by Eq. 17.9

$$\lambda_{\text{min}} \propto \frac{1}{V} \tag{17.9}$$

Using the fact that $\lambda\nu = c$ (Eq. 17.8), we can rewrite this result as,

$$\nu_{\text{max}} \propto V. \tag{17.10}$$

Although it is not relevant to our present discussion, we should point out, for the sake of completeness, that in addition to the continuous spectrum there are several intensity peaks, as shown in Fig. 17-8, called the ***characteristic X rays***. The wavelengths of the characteristic X rays are independent of the voltage V but depend on the material of the target.

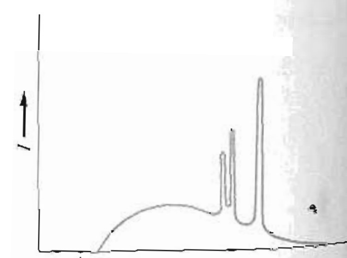


FIGURE 17-8 Intensity of X rays emitted versus the wavelength of the X rays. Note that no X rays of wavelength less than a critical value λ_{min} are emitted.

Let us try to understand the origin of the spectrum presented in Fig. 17-8. When the incoming electron enters the target, it will interact with the atomic electrons and with the nuclei present there. The interaction with the atomic electrons is the process that is primarily responsible for the slowing down of the incident electrons. Through multiple collisions, an incident electron is progressively slowed down and loses its energy to the target: The kinetic energy of the bombarding electron becomes heat. Occasionally, an electron-electron collision may occur, which results in a large transfer of energy from the incident electron to an atomic electron. As a result of this collision, the atomic electron will be knocked out of an atom. In subsequent chapters we will see that the electrons in the atom occupy discrete energy levels. When one of these energy levels is vacated as a result of a collision, one of the outer electrons in the

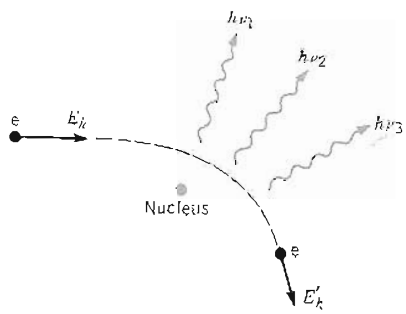


FIGURE 17-9 Schematic representation of an electron interacting with a nucleus in the target and emitting X rays in the process.

atom falls down into the vacated energy state and in the process gives off a photon. The energy of this photon $h\nu$, will be equal to the energy difference between the two atomic levels involved in the transition. These photons account for the characteristic X-ray peaks. Because, as will be shown in later chapters, the atomic energy levels are determined by the structure of the particular atom, we can understand why the frequency of the characteristic X rays depends on the target material and not on the energy of the incident electrons, that is, on the accelerating voltage V.

The incident electrons can also interact with the nuclei in the target. This interaction is responsible for the continuous spectrum or, its customary name, *bremsstrahlung* (braking radiation). Let us consider an electron with energy E_k approaching a positively charged nucleus. The electron, as a consequence of the coulombic attraction of the nucleus, will be deflected from its straight-line path; that is, it will be radially accelerated (see Fig. 17-9). Classical electromagnetic theory predicts that an accelerated charge will radiate electromagnetic waves *continuously and of all frequencies*. Thus, from the classical view of radiation, it is impossible to understand why there is a wavelength cut-off in the emission spectrum. The cut-off can be explained rather simply by the photon model of electromagnetic radiation. The accelerated electron will radiate energy not continuously but in quanta of energy $h\nu$. If now we consider the electron of Fig. 17-9 approaching the nucleus with energy E_k, emitting several photons of energy $h\nu_1$, $h\nu_2$, and so on, and finally leaving the nucleus with energy E'_k, we can, from energy conservation considerations, write

$$E_k - E'_k = h\nu_1 + h\nu_2 + \cdots$$

We have assumed that the nucleus does not acquire any energy during the collision. This is a good approximation because the nucleus is much heavier than the electron. It is now easy to understand the existence of the cut-off frequency. The most energetic photon that can be produced by the interaction of the electron with the nucleus is the one that is produced when the electron loses all its energy in the emission of a single photon. In such a case E'_k is zero and therefore

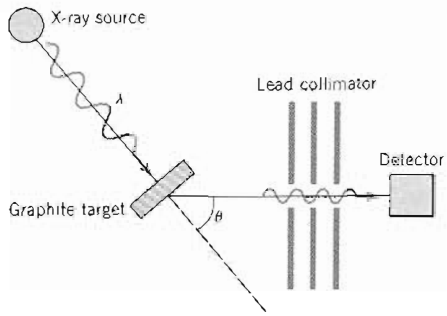


FIGURE 17-10 Apparatus used for the Compton effect experiment. Monochromatic X rays of wavelength λ are scattered by a graphite target. The wavel ength of the scattered X rays is determined for different angles θ.

$$E_k = h\nu_{max}$$

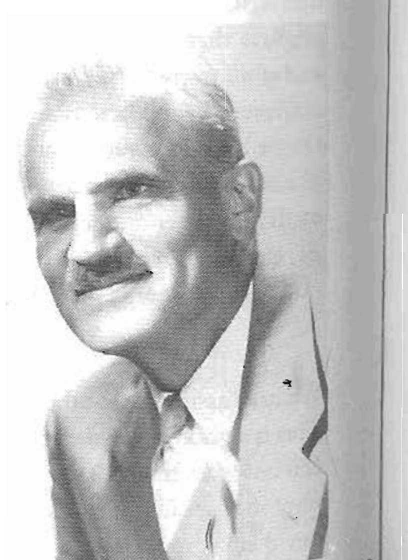
But because $E_k = eV$, we conclude that

$$\nu_{max} = \frac{e}{h}V \tag{17.11}$$

Equation 17.11 predicts the experimental observation given by Eq. 17.10. Not only can the photon model predict the existence of the frequency cut-off and its proper dependence on the accelerating voltage, but the experimental results can be used to determine the value of Planck's constant. The value of h obtained in this case agrees with the values obtained by Planck in connection with blackbody radiation and by Einstein in the case of the photoelectric effect.

17.4b. Compton Effect

In 1923, Arthur H. Compton (1892–1962) performed a series of experiments that provided a dramatic confirmation of the photon nature of electromagnetic radiation. A schematic of the experimental arrangement is shown in Fig. 17-10. A beam of X rays of sharply defined wavelength was sent onto a graphite target. Compton then studied the scattered radiation to see what wavelengths were present in it. This was done for different angles θ, between the incident and the scattered beams. Whereas the incident beam consisted of X rays of wavelength $\lambda = 0.709$ Å, Compton observed that the scattered beam contained two intensity maxima: one at $\lambda = 0.709$ Å and the other at a λ' greater than that of the incident radiation. The value of λ' increased as the angle of scattering increased and reached a maximum value of 0.758 Å for $\theta = 180°$. Figure 17-11 shows the intensity of the scattered radiation as a function of wavelength for four particular values of θ.



Arthur Holly Compton (1892–1962).

To understand how these results support the photon hypothesis, let us first consider what classical electromagnetic theory predicts about the scattering of electromagnetic waves. When an electric field $\mathcal{E} = \mathcal{E}_0 \sin(kx - \omega t)$ impinges on an electron in the target material, the electron will experience a force $\mathbf{F} = e\mathcal{E} = e\mathcal{E}_0 \sin(\omega t - kx)$, see Eq. 14.1. As a result of this force, the electron will oscillate with a frequency equal to that of the force, that is, the frequency of the incident electromagnetic wave. On the other hand, according to classical electromagnetic radiation theory, a charged particle (the electron in this case) undergoing simple harmonic motion radiates electromagnetic waves of the same frequency as the frequency of the motion of the charged particle (see Section 16.8). The electron plays the role of a transfer agent: It absorbs energy from the incident beam and reradiates this energy at the same frequency in all directions. Thus, classical electromagnetic theory cannot explain the presence of a longer wavelength (smaller frequency) in the scattered beam.

Compton explained the shift in the wavelength of the scattered beam by considering it to be a beam of photons, each with energy $E = h\nu$ and momentum $p = h\nu/c = h/\lambda$ (see the Supplement at the end of this chapter). According to Compton, the photons collide with the electrons in a particle-particle-like collision (see Fig. 17-12). In the collision, the electron will acquire some momentum and energy at the expense of the photon. From consideration of the conservation of energy we conclude that the energy of the photon and, hence, its frequency will decrease: the wavelengths of the scattered photons will be longer than that of the incident photons. Clearly, the stronger the collision, the larger the angle of scattering of the photon and the greater the energy lost to the electron; the shift in frequency should increase with increasing θ. In fact, we expect that the maximum energy transfer will occur in the case of head-on collision, which will result in backward scattering ($\theta = 180°$) of the photon. These arguments can be made quantitative by writing explicitly the conservation of energy and momentum equations. The solution of these equations, which can be found in most modern physics textbooks, yields the frequency and wavelength of the scattered photons as a function of θ. The result, originally derived by Compton, is

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta) \qquad (17.12)$$

where λ' and λ are, respectively, the wavelengths of the scattered and the incident photons; m is the mass of the scattering particle, the electron, and c is the velocity of light. An inspection of Eq. 17.12 corroborates the qualitative arguments presented earlier. In particular, $\lambda' - \lambda$ is a maximum when $\cos\theta = -1$, that is, when $\theta = 180°$. This maximum shift in wavelength is therefore

$$(\lambda' - \lambda)_{\max} = \frac{2h}{mc} = \frac{2 \times 6.63 \times 10^{-34}\ \text{J-sec}}{9.1 \times 10^{-31}\ \text{kg} \times 3 \times 10^{8}\ \text{m/sec}}$$

$$= 0.049 \times 10^{-10}\ \text{m} = 0.049\ \text{Å}$$

For $\theta = 90°$

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos 90°)$$

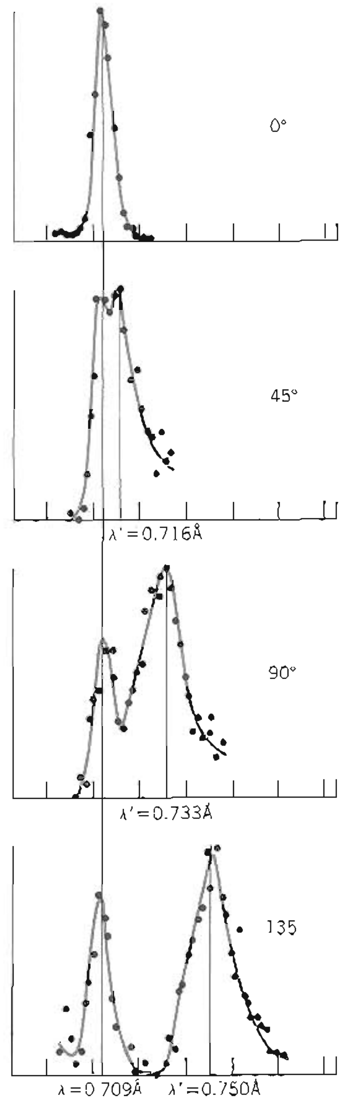


FIGURE 17-11 Intensity of the scattered X rays in the Compton experiment of Fig. 17-10 as a function of the wavelength for four different angles θ. (*Source*: Kenneth Krane, *Modern Physics*. Copyright ©1983 by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.)

$$= 0.709 \times 10^{-10}\,\text{m}$$

$$+ \frac{6.63 \times 10^{-34}\ \text{J-sec}}{9.1 \times 10^{-31}\ \text{kg} \times 3 \times 10^8\ \text{m/sec}}(1 - 0)$$

$$= 0.733 \times 10^{-10}\,\text{m} = 0.733\,\text{Å}$$

which agrees with the results shown in Fig. 17-11c.

The unshifted wavelength present in the scattered beam can also be explained in terms of the photon model. Up to now, we have assumed that the photons collide with the electrons in the target material. The photons can also collide with the atoms in the graphite. In this case, the only change to be made in the calculations consists in replacing in Eq. 17.12 the mass of the electron by that of the carbon atom. For graphite, $m_{\text{atom}} \approx 24,000 m_{\text{electron}}$. From Eq. 17.12, we can see that the shift in wavelength will be 24,000 times smaller, that is,

$$(\lambda' - \lambda)_{\max} \approx \frac{0.049\,\text{Å}}{24,000} = 2 \times 10^{-6}\,\text{Å}$$

This is an insignificant and unobservable amount when we compare it with $\lambda = 0.709\,\text{Å}$.

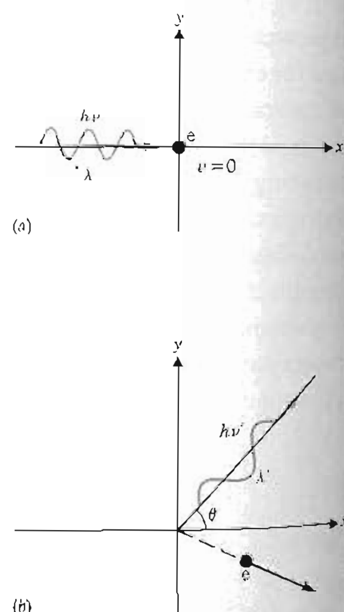


FIGURE 17-12 Scattering process of the X rays in the Compton experiment of Fig. 17-10. A photon collides with an electron in the graphite target in a particle-particle-like collision. The photon imparts some energy to the electron, resulting in a decrease in its own energy (and therefore in its frequency) and a concommitant increase in its wavelength. (*a*) Before the collision. (*b*) After the collision.

EXAMPLE 17-3

X rays of wavelength $\lambda = 0.700\,\text{Å}$ are Compton-scattered by the electrons in a graphite target. (a) What is the wavelength of the X rays scattered at an angle $\theta = 120°$? (b) What is the kinetic energy of the scattering electrons if they were originally at rest? (c) What is the scattering angle of the electrons?

Solution

(a) From Eq. 17.12

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos 120°)$$

$$= 0.700 \times 10^{-10}\,\text{m}$$

$$+ \frac{6.63 \times 10^{-34}\ \text{J-sec}}{9.1 \times 10^{-31}\ \text{kg} \times 3 \times 10^8\ \text{m/sec}}(1 + 0.5)$$

$$= 0.736 \times 10^{-10}\,\text{m} = 0.736\,\text{Å}$$

(b) From conservation of energy principles, the kinetic energy of the electron is equal to the energy lost by the photon, that is,

$$E_k = h\nu - h\nu'$$

which from Eq. 17.8, $\lambda \nu = c$, can be written as

$$E_k = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$= 6.63 \times 10^{-34}\ \text{J-sec}$$

$$+ \times 3 \times 10^{8}\ \text{m/sec}\left[\frac{1}{0.700 \times 10^{-10}\ \text{m}} - \frac{1}{0.736 \times 10^{-10}\ \text{m}}\right]$$

$$= 1.39 \times 10^{-16}\ \text{J} = 869\ \text{eV}$$

(c) From conservation of linear momentum principles

$$p_x(\text{before the collision}) = p_x(\text{after the collision})$$

Writing this explicitly (see Fig. 17-13 and Eq. 17.18 in Supplement 17-1).

$$\frac{h\nu}{c} = p_e \cos\phi - \frac{h\nu'}{c}\cos 60^\circ$$

therefore

$$p_e \cos\phi = \frac{h\nu}{c} + \frac{h\nu'}{c}\cos 60^\circ$$

$$= \frac{h}{\lambda} + \frac{h}{\lambda'}\cos 60^\circ$$

$$p_e \cos\phi = 6.63 \times 10^{-34}\ \text{J-sec}\left[\frac{1}{0.700 \times 10^{-10}\ \text{m}} + \frac{0.5}{0.736 \times 10^{-10}\ \text{m}}\right]$$

$$p_e \cos\phi = 13.98 \times 10^{-24}\ \text{kg m/sec} \qquad (17.13)$$

Again from conservation of momentum,

$$p_y(\text{before the collision}) = p_y(\text{after the collision})$$

$$0 = -p_e \sin\phi + \frac{h\nu'}{c}\sin 60^\circ$$

Therefore

$$p_e \sin\phi = \frac{h\nu'}{c}\sin 60^\circ = \frac{h}{\lambda'}\sin 60^\circ$$

$$p_e \sin\phi = \frac{6.63 \times 10^{-34}\ \text{J-sec}}{0.736 \times 10^{-10}\ \text{m}} \times (0.866)$$

$$p_e \sin\phi = 7.80 \times 10^{-24}\ \text{kg m/sec} \qquad (17.14)$$

Combining Eqs. 17.13 and 17.14, we obtain

$$\tan\phi = \frac{7.80 \times 10^{-24}\ \text{kg m/sec}}{13.98 \times 10^{-24}\ \text{kg m/sec}} = 0.56$$

$$\phi = 29.2^\circ$$

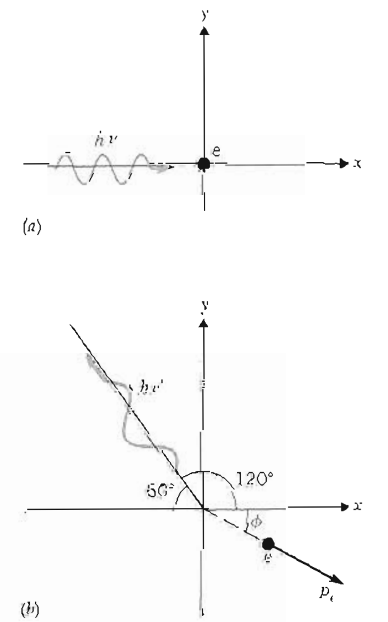


FIGURE Figure17-13 Example 17-3. (a) Before the collision. (b) After the collision.

SUPPLEMENT 17-1: MOMENTUM OF THE PHOTON

From Einstein's theory of special relativity, the total relativistic energy of a particle is given by

$$E = E_k + E_0$$

$$E = mc^2 \qquad (17.15)$$

where E_k is the kinetic energy of the particle, $E_0 = m_0c^2$ is the rest energy (m_0 is the rest mass of the particle, that is, the mass of the particle when its velocity is zero), m is the relativistic mass (the mass when the particle is moving), and c is the velocity of light. For the photon, the rest energy and therefore the rest mass is zero; a photon at rest does not exist.

We can now use Einstein's postulate, namely, $E_{photon} = h\nu$ and, combining this with the expression for the total relativistic energy, Eq. 17.15, we can get an expression for the momentum of the photon.

$$E_{photon} = h\nu = mc^2$$

Dividing by c, we obtain

$$mc = \frac{h\nu}{c} \qquad (17.16)$$

Using Eq. 17.8, we obtain

$$mc = \frac{h}{\lambda} \qquad (17.17)$$

We recognize the left side of Eqs. 17.16 and 17.17 as the momentum p of the photon, that is, the product of the mass and the velocity of the photon,

$$p_{photon} = \frac{h}{\lambda} = \frac{h\nu}{c} \qquad (17.18)$$

PROBLEMS

17.1 The wavelength λ_{max} for which the spectral radiancy, I, of a blackbody is a maximum, is given by Wien's displacement law, which states:

$$\lambda_{max} T = 2.898 \times 10^{-3} \text{ m-K}$$

where T is the absolute temperature of the blackbody. The temperature of the surface of the sun is roughly 5800 K. (a) What is the wavelength of the most intense radiation emitted by the sun? (b) In what part of the electromagnetic spectrum is this radiation?

17.2 The rate at which the sun's energy strikes the earth is known as the solar constant and has a value of 2 cal/cm^2-sec. If we assume that there is no reflection by the clouds, ice caps, or such, the earth would absorb all this energy. The earth would be a blackbody radiator and would radiate all the energy that it receives back into space. What would be the equilibrium temperature of the earth if our assumption were true?

Answer: 1100 K.

17.3 The radius of a hydrogen atom is approximately 10^{-10} m (1 Å). Light of intensity 1.0 W/m^2 is shone on such an atom. What is the time lag for the photoelectric effect on the basis of the wave theory of light? The binding energy of the electron in the hydrogen atom is 13.6 eV.

Answer: 69.3 sec.

17.4 A 5000-W radio transmitter emits radiation of frequency $\nu = 1100$ kHz. How many photons per second does it emit?

17.5 Consider a 100-W sodium vapor lamp radiating energy uniformly in all directions. Assume that 80% of the energy radiated is in the form of photons of wavelength 5890 Å. (a) What is the rate of photon emission by the lamp? (b) How far from the lamp will the average density of photons be 2 photons/cm^2-sec? (c) What is the photon flux (that is, the number of photons per unit time per unit area) 2.0 m from the lamp?

Answer: (a) 2.37×10^{20} sec^{-1}, (b) 3.07×10^{7} m, (c) 4.71×10^{14} sec^{-1} cm^{-2}.

17.6 For a quick estimate of the energy of a photon in eV, physicists use the relation $E(\text{eV}) = 12,345/\lambda(\AA)$. By what percentage is this relation inaccurate?

17.7 The basis for the creation of the latent image on a photographic negative is the dissociation of molecules of silver bromide, AgBr. The heat of dissociation of AgBr is 24 kcal/mole. Find the longest wavelength of light that is just able to expose the negative, that is, dissociate AgBr.

Answer: 11,900 Å.

17.8 A metal has a work function $\phi = 1.5$ eV. (a) What is the stopping potential for light of wavelength 3000 Å? (b) What is the maximum velocity of the emitted photoelectrons?

17.9 Light of wavelength 1500 Å falls on an aluminum surface having a work function of 4.2 eV. (a) What is the kinetic energy of the fastest emitted photoelectrons? (b) What is the stopping potential? (c) What is the cut-off frequency ν_c for aluminum?

Answer: (a) 4.09 eV, (b) 4.09 V, (c) 1.01×10^{15} Hz.

17.10 When light of wavelength 2000 Å is incident on the surface of a metal, the electrons are emitted with a maximum kinetic energy of 2.0 eV. (a) Calculate the energy of the incident photons. (b) What is the work function of the metal? (c) If the incident light had a wavelength of 6000 Å, what would be the stopping potential?

17.11 The cut-off frequency for photoemission for a given metal is ν_0. What is the maximum energy of the emitted electrons when the metal is illuminated with light of frequency $3\nu_0$?

17.12 When light of frequency ν_0 is incident on a certain metal surface electrons are emitted with a maximum kinetic energy of 15 eV. When the frequency is reduced to $\nu_0/2$, the maximum kinetic energy is 3 eV. What is the cut-off frequency for photoemission for this metal?

Answer: 2.17×10^{15} Hz.

17.13 In a photoelectric effect experiment, it is found that when the surface of sodium metal is illuminated with light of wavelength $\lambda = 4200$ Å, the stopping potential $V_0 = 0.65$ V. When the metal is illuminated with light of wavelength $\lambda = 3100$ Å, the stopping potential is $V_0 = 1.69$ V. Calculate Planck's constant from these data.

17.14 Not every photon striking the surface of a metal undergoes a collision with the electrons in the metal. An important quantity in the extension of the photoelectric effect theory is the quantum efficiency, namely, how many photons are required on the average to yield one photoelectron. In a typical experiment, light of wavelength $\lambda = 4366$ Å is shone on a potassium surface. The observed yield is 8×10^{-3} C/J. How many photons are required to yield one photoelectron?

Answer: 44 photons.

17.15 Electrons in an X-ray tube are accelerated through a potential difference of 5000 V. What is the maximum frequency and the minimum wavelength of the X rays produced?

17.16 Alpha particles (charge $+2e$) are accelerated by an electric potential difference of 20,000 V. The α particles strike a metal target and in the process produce X rays. Find the smallest wavelength of the X rays emitted by the target.

Answer: 0.311 Å.

17.17 A photon of frequency $\nu = 3 \times 10^{18}$ Hz is Compton-scattered by an electron initially at rest. After the collision, the electron moves in the direction of the incident photon. (a) Find the wavelength of the scattered photon. (b) What is the energy of the scattered electron?

17.18 X rays of wavelength 1 Å are Compton-scattered by the electrons in a carbon target. (a) Calculate the wavelength of the X rays scattered at 90° with respect to the incident X rays. (b) What is the energy of the electrons causing the scattering?

17.19 In a Compton experiment the wavelength of the incident photon is 1.3249 Å, whereas that of the scattered photon is 1.3461 Å. (a) At what angle is the photon scattered? (b) At what angle is the electron scattered? (c) What is the kinetic energy of the scattered electron?

Answer: (a) 82.7°, (b) 48.1°, (c) 148 eV.