THE DIFFERENTIAL GEOMETRY OF PARAMETRIC PRIMITIVES

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Abstract: We derive the expressions for first and second derivatives, normal, metric matrix and curvature matrix for spheres, cones, cylinders, and tori.

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Differential Properties of Parametric Surfaces

A parametric surface is a function:

$$\mathbf{x} = \mathbf{F}(\mathbf{u})$$

where

$$\mathbf{x} = [x \quad y \quad z]$$

is a point in affine 3-space, and

$$\mathbf{u} = [u \quad v]$$

is a point in affine 2-space.

The *Jacobian* matrix is a matrix of partial derivatives that relate changes in *u* and *v* to changes in *x*, *y*, and *z*:

$$\mathbf{J} = \frac{\partial(x,y,z)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial u} \\ \frac{\partial \mathbf{x}}{\partial v} \end{bmatrix}$$

The Hessian is a tensor of second partial derivatives:

$$\mathbf{H} = \frac{\partial^2(x,y,z)}{\partial(u,v)\partial(u,v)} = \begin{bmatrix} \begin{bmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \end{bmatrix} & \begin{bmatrix} \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial^2 y}{\partial u \partial v} & \frac{\partial^2 z}{\partial u \partial v} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial^2 x}{\partial v \partial u} & \frac{\partial^2 y}{\partial v \partial u} & \frac{\partial^2 z}{\partial v \partial u} \end{bmatrix} & \begin{bmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 \mathbf{x}}{\partial u^2} & \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \\ \frac{\partial^2 \mathbf{x}}{\partial v \partial u} & \frac{\partial^2 \mathbf{x}}{\partial v^2} \end{bmatrix}$$

The first fundamental form is defined as:

$$\mathbf{G} = \mathbf{J}\mathbf{J}^t = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v} \\ \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial v} \end{bmatrix}$$

and establishes a metric of differential length:

$$(d\mathbf{x})^2 = (d\mathbf{u})\mathbf{G}(d\mathbf{u})^t$$

so that the arc length of a curve segment, $\mathbf{u} = \mathbf{u}(t), \quad t_0 < t < t_1$ is given by:

$$s = \int_{t_0}^{t_1} \frac{ds}{dt} dt = \int_{t_0}^{t_1} |\dot{\mathbf{x}}| dt = \int_{t_0}^{t_1} |\dot{\mathbf{x}}| dt = \int_{t_0}^{t_1} (\dot{\mathbf{u}}\mathbf{G}\dot{\mathbf{u}}^t)^{\frac{1}{2}} dt$$

The differential surface area enclosed by the differential parallelogram $(\delta u, \delta v)$ is approximately:

$$\delta S \approx (|\mathbf{G}|)^{\frac{1}{2}} \delta u \delta v$$

so that the area of a region of the surface corresponding to a region *R* in the *u*-*v* plane is:

$$S = \iint_R (|\mathbf{G}|)^{\frac{1}{2}} du dv$$

The second fundamental matrix measures normal curvature, and is given by:

$$\mathbf{D} = \mathbf{n} \cdot \mathbf{H} = \begin{bmatrix} \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial u^2} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \\ \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial v \partial u} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial v^2} \end{bmatrix}$$

The normal curvature is defined to be positive a curve $\mathbf{u}$ on the surface turns toward the positive direction of the surface normal by:

$$\kappa_n = \frac{\dot{\mathbf{u}}\mathbf{D}\dot{\mathbf{u}}^t}{\dot{\mathbf{u}}\mathbf{G}\dot{\mathbf{u}}^t}$$

The deviation (in the normal direction) from the tangent plane of the surface, given a differential displacement of $\dot{\mathbf{u}}$ is:

$$\ddot{\mathbf{x}} \cdot \mathbf{n} = \dot{\mathbf{u}}\mathbf{D}\dot{\mathbf{u}}^t$$

Reparametrization

If the parametrization of the surface is transformed by the equations:

$$u' = u'(u,v) \quad \text{and} \quad v' = v'(u,v)$$

then the chain rule yields:

$$\frac{\partial(x,y,z)}{\partial(u',v')} = \frac{\partial(u,v)}{\partial(u',v')}\frac{\partial(x,y,z)}{\partial(u,v)}$$

or

$$\mathbf{J}' = \mathbf{PJ}$$

where

$$\mathbf{J}' = \frac{\partial(x,y,z)}{\partial(u',v')}$$

is the new Jacobian matrix of the surface with respect to the new parameters u' and v', and

$$\mathbf{P} = \frac{\partial(u,v)}{\partial(u',v')} = \begin{bmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{bmatrix}$$

is the Jacobian matrix of the reparametrization.

The new Hessian is given by

$$\mathbf{H}' = \mathbf{PHP}^T + \mathbf{QJ}$$

where

$$\mathbf{Q} = \begin{bmatrix} \frac{\partial(u,v)}{\partial u'^2} & \frac{\partial(u,v)}{\partial u' \partial v'} \\ \frac{\partial(u,v)}{\partial v' \partial u'} & \frac{\partial(u,v)}{\partial v'^2} \end{bmatrix}.$$

The new fundamental matrix is given by:

$$\mathbf{G}' = \mathbf{PGP}^T$$

and the new curvature matrix is given by:

$$\mathbf{D}' = \mathbf{PDP}^T$$

Change of Coordinates

For simplicity, we have defined several primitives with unit size, located at the origin. Related to the reparametrization is the change of coordinates $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, with associated Jacobian:

$$\mathbf{C} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{bmatrix}$$

When the change of coordinates is represented by the affine transformation:

$$\mathbf{A} = \begin{bmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \\ x_o & y_o & z_o \end{bmatrix}$$

the Jacobian is simply the submatrix:

$$\mathbf{C} = \begin{bmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{bmatrix}$$

Regardless, the Jacobian and Hessian transform as follows:

$$\mathbf{J}' = \mathbf{JC}, \quad \mathbf{H}' = \mathbf{HC}$$

The normal is transformed as:

$$\mathbf{n}' = \frac{\mathbf{nC}^{-1t}}{\left(\mathbf{nC}^{-1t}\mathbf{C}^{-1}\mathbf{n}^t\right)^{\frac{1}{2}}}$$

The denominator arises from the desire to have a *unit* normal.

The first and second fundamental matrices are then calculated as:

$$\mathbf{G}' = \mathbf{J}'\mathbf{J}'^t = \mathbf{JCC}^t\mathbf{J}^t$$

$$\mathbf{D}' = \mathbf{H}' \cdot \mathbf{n}' = \frac{(\mathbf{HC}) \cdot \left(\mathbf{nC}^{-1t}\right)}{\left(\mathbf{nC}^{-1t}\mathbf{C}^{-1}\mathbf{n}^t\right)^{\frac{1}{2}}} = \frac{\mathbf{HCC}^{-1}\mathbf{n}^t}{\left(\mathbf{nC}^{-1t}\mathbf{C}^{-1}\mathbf{n}^t\right)^{\frac{1}{2}}} = \frac{\mathbf{H} \cdot \mathbf{n}}{\left(\mathbf{nC}^{-1t}\mathbf{C}^{-1}\mathbf{n}^t\right)^{\frac{1}{2}}} = \frac{\mathbf{D}}{\left(\mathbf{nC}^{-1t}\mathbf{C}^{-1}\mathbf{n}^t\right)^{\frac{1}{2}}}$$

Not very pretty. But certain types of transformations can be applied easily. For a uniform scale with arbitrary translations,

$$\mathbf{C} = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} = r\,\mathbf{I}$$

so that

$$\mathbf{J}' = r\mathbf{J}, \quad \mathbf{H}' = r\mathbf{H}, \quad \mathbf{n}' = \mathbf{n}, \quad \mathbf{G}' = r^2\mathbf{G}, \quad \mathbf{D}' = r\mathbf{D}$$

For rotations (and arbitrary translations), the Jacobian matrix $\mathbf{C}=\mathbf{R}$ is orthogonal, so the inverse is equal to the transpose, yielding:

$$\mathbf{J}' = \mathbf{JR}, \quad \mathbf{H}' = \mathbf{HR}, \quad \mathbf{n}' = \mathbf{nR}, \quad \mathbf{G}' = \mathbf{G}, \quad \mathbf{D}' = \mathbf{D}$$

Combining the two, we have the results for a transformation that includes translations, rotations and uniform scale:

$$\mathbf{J}' = r\mathbf{JR}, \quad \mathbf{H}' = r\mathbf{HR}, \quad \mathbf{n}' = \mathbf{nR}, \quad \mathbf{G}' = r^2\mathbf{G}, \quad \mathbf{D}' = r\mathbf{D}$$

or in terms of the composite matrix $\mathbf{C} = r\,\mathbf{R}$:

$$\mathbf{J}' = \mathbf{JC}, \quad \mathbf{H}' = \mathbf{HC}, \quad \mathbf{n}' = \frac{\mathbf{nC}}{\left(|\mathbf{C}|\right)^{\frac{1}{3}}}, \quad \mathbf{G}' = \left(|\mathbf{C}|\right)^{\frac{2}{3}}\mathbf{G}, \quad \mathbf{D}' = \left(|\mathbf{C}|\right)^{\frac{1}{3}}\mathbf{D}$$

Sphere

Given the spherical coordinates:

$$[x \quad y \quad z] = [r \sin\varphi \cos\theta \quad r \sin\varphi \sin\theta \quad r \cos\varphi]$$

we have the Jacobian matrix:

$$\frac{\partial(x,y,z)}{\partial(\theta,\varphi)} = \begin{bmatrix} -y & x & 0 \\ \frac{xz}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}} & -\sqrt{x^2+y^2} \end{bmatrix}$$

the Hessian tensor:

$$\frac{\partial^2(x,y,z)}{\partial(\theta,\varphi)\partial(\theta,\varphi)} = \begin{bmatrix} [-x \quad -y \quad 0] & \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} & [-x \quad -y \quad -z] \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D} = \begin{bmatrix} -\frac{x^2+y^2}{r} & 0 \\ 0 & -r \end{bmatrix}$$

Unit Sphere

Angle Parametrization

Given the unit spherical coordinates with $0 \le \theta < 2\pi$, $0 \le \phi < \pi$, we parametrize the sphere:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \end{bmatrix}$$

This yields the Jacobian matrix:

$$\mathbf{J}_{\theta\varphi} = \begin{bmatrix} -y & x & 0 \\ \frac{xz}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}} & -\sqrt{x^2+y^2} \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{\theta\varphi} = \begin{bmatrix} \begin{bmatrix} -x & -y & 0 \end{bmatrix} & \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} & \begin{bmatrix} -x & -y & -z \end{bmatrix} \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{\theta\varphi} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & 1 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \begin{bmatrix} x & y & z \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{\theta\varphi} = \begin{bmatrix} -\left(x^2+y^2\right) & 0 \\ 0 & -1 \end{bmatrix}$$

Angle Parametrization

With the reparametrization $\theta = 2\pi u$, $\phi = \pi v$, we have the Jacobian:

$$\mathbf{P} = \begin{bmatrix} 2\pi & 0 \\ 0 & \pi \end{bmatrix}$$

Applying the chain rule, we have:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2\pi y & 2\pi x & 0 \\ \frac{\pi xz}{\sqrt{x^2+y^2}} & \frac{\pi yz}{\sqrt{x^2+y^2}} & -\pi\sqrt{x^2+y^2} \end{bmatrix}$$

$$\mathbf{H}_{uv} = \begin{bmatrix} 4\pi^2\begin{bmatrix} -x & -y & 0 \end{bmatrix} & 2\pi\begin{bmatrix} -\dfrac{yz}{\sqrt{x^2+y^2}} & \dfrac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ 2\pi\begin{bmatrix} -\dfrac{yz}{\sqrt{x^2+y^2}} & \dfrac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} & \pi^2\begin{bmatrix} -x & -y & -z \end{bmatrix} \end{bmatrix}$$

$$\mathbf{G}_{uv} = \begin{bmatrix} 4\pi^2\left(x^2+y^2\right) & 0 \\ 0 & \pi^2 \end{bmatrix}$$

$$\mathbf{D}_{uv} = \begin{bmatrix} -4\pi^2\left(x^2+y^2\right) & 0 \\ 0 & -\pi^2 \end{bmatrix}$$

Changing coordinates to yield a sphere of arbitrary radius, we find that the expressions for the Jacobian, the Hessian, and the metric matrix remain the same, because x, y, and z scale linearly with r. The curvature matrix changes to:

$$\mathbf{D}_{uv} = \begin{bmatrix} -\dfrac{4\pi^2\left(x^2+y^2\right)}{r} & 0 \\ 0 & -\pi^2 r \end{bmatrix}$$

Cone

Angle Parametrization

Given the unit conical parametrization:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} z\cos\theta & z\sin\theta & z \end{bmatrix}$$

we have the Jacobian matrix:

$$\mathbf{J}_{\theta z} = \begin{bmatrix} -y & x & 0 \\ \frac{x}{z} & \frac{y}{z} & 1 \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{\theta z} = \begin{bmatrix} \begin{bmatrix} -x & -y & 0 \end{bmatrix} & \begin{bmatrix} -\frac{y}{z} & \frac{x}{z} & 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{y}{z} & \frac{x}{z} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{\theta z} = \begin{bmatrix} x^2 + y^2 & 0 \\ 0 & \frac{x^2 + y^2 + z^2}{z^2} \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ 0 & 2 \end{bmatrix}$$

the normal:

$$\mathbf{n}_{\theta z} = \begin{bmatrix} \frac{x}{z\sqrt{2}} & \frac{y}{z\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{\theta z} = \begin{bmatrix} -\frac{z}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Parametrization

For the parametrization:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} rv\cos 2\pi u & rv\sin 2\pi u & vh \end{bmatrix}$$

we have:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2\pi y & 2\pi x & 0 \\ \frac{hx}{rz} & \frac{hy}{rz} & h \end{bmatrix}$$

$$\mathbf{H}_{uv} = \begin{bmatrix} 4\pi^2 \begin{bmatrix} -x & -y & 0 \end{bmatrix} & \frac{2\pi h}{rz} \begin{bmatrix} -y & x & 0 \end{bmatrix} \\ \frac{2\pi h}{rz} \begin{bmatrix} -y & x & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\mathbf{G}_{uv} = \begin{bmatrix} 4\pi^2\left(x^2 + y^2\right) & 0 \\ 0 & \dfrac{h^2\left(x^2 + y^2 + z^2\right)}{z^2} \end{bmatrix}$$

$$\mathbf{n}_{uv} = \frac{1}{\sqrt{1+h^2}} \begin{bmatrix} \dfrac{h^2 x}{rz} & \dfrac{h^2 y}{rz} & -1 \end{bmatrix}$$

$$\mathbf{D}_{uv} = \begin{bmatrix} -\dfrac{4\pi^2 rz}{\sqrt{1+h^2}} & 0 \\ 0 & 0 \end{bmatrix}$$

Cylinder

Angle Parametrization

Given the cylindrical parametrization:

$$[x \quad y \quad z] = [\cos\theta \quad \sin\theta \quad z]$$

we have the Jacobian matrix:

$$\mathbf{J}_{\theta\varphi} = \begin{bmatrix} -y & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{\theta\varphi} = \begin{bmatrix} [-x \quad -y \quad 0] & [0 \quad 0 \quad 0] \\ [0 \quad 0 \quad 0] & [0 \quad 0 \quad 0] \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{\theta\varphi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the normal:

$$\mathbf{n} = [x \quad y \quad 0]$$

and the second fundamental form:

$$\mathbf{D}_{\theta\varphi} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Parametrization

With the parametrization:

$$[x \quad y \quad z] = [r\cos 2\pi u \quad r\sin 2\pi u \quad hv]$$

we have the Jacobian matrix:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2\pi y & 2\pi x & 0 \\ 0 & 0 & h \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{uv} = \begin{bmatrix} [-4\pi^2 x \quad -4\pi^2 y \quad 0] & [0 \quad 0 \quad 0] \\ [0 \quad 0 \quad 0] & [0 \quad 0 \quad 0] \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{uv} = \begin{bmatrix} 4\pi^2 r^2 & 0 \\ 0 & h^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} & 0 \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{uv} = \begin{bmatrix} -4\pi^2 r & 0 \\ 0 & 0 \end{bmatrix}$$

Torus

Angle Parametrization

Given the torus parametrization:

$$[x \quad y \quad z] = [(R + r\cos\varphi)\cos\theta \quad (R + r\cos\varphi)\sin\theta \quad r\sin\varphi]$$

we have the Jacobian matrix:

$$\mathbf{J}_{\theta\varphi} = \begin{bmatrix} -y & x & 0 \\ -\dfrac{xz}{\sqrt{x^2+y^2}} & -\dfrac{yz}{\sqrt{x^2+y^2}} & \sqrt{x^2+y^2} - R \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{\theta\varphi} = \begin{bmatrix} [-x \quad -y \quad 0] & \begin{bmatrix} \dfrac{yz}{\sqrt{x^2+y^2}} & -\dfrac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ \begin{bmatrix} \dfrac{yz}{\sqrt{x^2+y^2}} & -\dfrac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} & \begin{bmatrix} -x\left(1 - \dfrac{R}{\sqrt{x^2+y^2}}\right) & -y\left(1 - \dfrac{R}{\sqrt{x^2+y^2}}\right) & -z \end{bmatrix} \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{\theta\varphi} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \begin{bmatrix} x\dfrac{1 - \dfrac{R}{\sqrt{x^2+y^2}}}{r} & y\dfrac{1 - \dfrac{R}{\sqrt{x^2+y^2}}}{r} & \dfrac{z}{r} \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{\theta\varphi} = \begin{bmatrix} -\dfrac{x^2+y^2}{r}\left(1 - \dfrac{R}{\sqrt{x^2+y^2}}\right) & 0 \\ 0 & -r \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{R^2 - x^2 - y^2 + z^2 - r^2}{2r} & 0 \\ 0 & -r \end{bmatrix}$$

using the torus's implicit equation:

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2$$