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The Theory of Riemann Integration¹

1 The Integral

Through the work on calculus, particularly integration, and its application throughout the 18th century was formidable, there was no actual “theory” for it.

The applications of calculus to problems of physics, i.e. partial differential equations, and the fledgling ideas of function representation by trigonometric series required clarification of just what a function was. Correspondingly, this challenged the notion that an integral is just an antiderivative.

Let’s trace this development of the integral as a rough and ready way to solve problems of physics to a full-fledged theory.

We begin the story with sequence of events.....

1. **Leonhard Euler** (1707-1783) and **Jean d’Alembert** (1717-1783) argue in 1730-1750’s over the “type” of solutions that should be admitted as solutions to the wave equation

$$u_{xy} = 0$$

D’Alembert showed that a solution must have the form

$$F(x, t) = \frac{1}{2}[f(x + t) + f(x - t)].$$

For $t = 0$ we have the initial shape $f(x)$.

Note: Here a function is just that. The new notation and designation are fixed.

But just what kinds of functions f can be admitted?

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2. D'Alembert argued f must be “continuous”, i.e. given by a single equation. Euler argued the restriction to be unnecessary and that f could be “discontinuous”, i.e. it could be formed of many curves.

In the modern sense though both are continuous.

3. **Daniel Bernoulli** (1700-1782) entered the fray by announcing that solutions must be expressible in a series of the form

$$f(x) = a_1 \sin(\pi x/L) + a_2 \sin(2\pi x/L) + \dots,$$

where L is the length of the string.

Euler, d'Alembert and **Joseph Lagrange** (1736-1813) strongly reject this.

4. In the 19th century the notion of arbitrary function again took center stage when Joseph Fourier (1768-1830) presented his celebrated paper² on heat conduction to the Paris Academy (1807). In its most general form, Fourier's proposition states:

Any (bounded) function f defined on $(-a, a)$ can be expressed as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x/a + b_n \sin n\pi x/a,$$

where

$$\begin{aligned} a_0 &= \frac{2}{a} \int_{-a}^a f(x) dx \\ a_n &= \frac{1}{a} \int_{-a}^a f(x) \cos n\pi x/a dx, \\ b_n &= \frac{1}{a} \int_{-a}^a f(x) \sin n\pi x/a dx. \end{aligned}$$

5. For Fourier the notion of function was rooted in the 18th century. In spite of the generality of his statements a “general” function for him was still continuous in the modern sense. For example, he would call

$$f(x) = \begin{cases} e^{-x} & x < 0 \\ e^x & x \geq 0 \end{cases}$$

²This work remained unpublished until 1822.

discontinuous.

6. Fourier believed that arbitrary functions behaved very well, that any $f(x)$ must have the form

$$f(x) = \frac{1}{2\pi} \int_a^b f(\alpha) d\alpha \int_{-\infty}^{\infty} \cos(px - p\alpha) dp,$$

which is of course meaningless.

7. For Fourier, a general function was one whose graph is smooth except for a finite number of exceptional points.

8. Fourier believed and attempted to validate that if the coefficients $a_1 \dots, b_1 \dots$ could be determined then the representation must be valid.

His original proof involved a power series representation and some manipulations with an infinite system of equations.

Lagrange improved things using a more modern appearing argument:

- (a) multiply by $\cos n\pi/a$,
- (b) integrate between $-a$ and a , term-by-term,
- (c) interchange $\int_{-a}^a \sum_1^{\infty}$ to $\sum_1^{\infty} \int_{-a}^a$. With the “orthogonality” of the trig functions the Fourier coefficients are achieved.

The interchange $\int \Sigma$ to $\Sigma \int$ was not challenged until 1826 by **Niels Henrik Abel** (1802-1829).

The validity of term-by-term integration was lacking until until Cauchy proved conditions for it to hold.

Nonetheless, even granting the Lagrange program, the points were still thought to be lacking validity until **Henri Lebesgue** (1875-1941) gave a proper definition of area from which these issues are simple consequences.

9. Gradually, the integral becomes area based rather than antiderivative based. Thus area is again geometrically oriented. Remember though.....

Area is not yet properly defined.

And this issue is to become central to the concept of integral.

10. It is **Augustin Cauchy** (1789-1857) who gave us the modern definition of continuity and defined the definite integral as a limit of a sum. He began this work in 1814.

11. In his *Cours d'analyse* (1821) he gives the modern definition of continuity at a point (but uses it over an interval). Two years later he defines the limit of the Cauchy sum

$$\sum_{i=1}^n f(x_i)(x_i - x_{i-1})$$

as the definite integral for a continuous function. Moreover, he showed that for any two partitions, the sums could be made arbitrarily small provided the norms of the partitions are sufficiently small. By taking the limit, Cauchy obtains the definite integral.

The basic refinement argument is this: For continuous (i.e. uniformly continuous) functions, the difference of sums S_P and $S_{P'}$ can be made arbitrarily small as a function of the maximum of the norms of the partitions.

This allowed Cauchy to consider **primitive** functions,

$$F(x) := \int_a^x f$$

He proved:

- **Theorem I.** F is a primitive function; that is $F' = f$
- **Theorem II.** All primitive functions have the form $\int_a^x f + C$.

To prove Theorem II he required

- **Theorem III.** If G is a function such that $G'(x) = 0$ for all x in $[a, b]$, then $G(x)$ remains constant there.

Theorems I, II and III form the Fundamental Theorem of Calculus. The proof depends on the *then* remarkable results about partition refinement. Here he (perhaps unwittingly) invokes uniform continuity.

12. Nonetheless Cauchy still regards functions as equations, that is $y = f(x)$ or $f(x, y) = 0$.

13. *Real* discontinuous functions finally emerge as those having the form

$$f(x) = \sum_{r=1}^n \chi_{I_r}(x)g_r(x),$$

where $\{I_r\}$ is a partition of $[a, b]$ and each $g_r(x)$ is a continuous (18th century) function on I_r . Cauchy's theory works for such functions with suitable adjustments. For this notion, the meaning of Fourier's a_n and b_n is resolved.

14. **Peter Gustav Lejeune-Dirichlet** (1805-1859) was the first mathematician to call attention to the existence of functions discontinuous at an infinite number of points. He gave the first rigorous proof of convergence of Fourier series under general conditions by considering partial sums³

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^n a_n \cos kx + b_n \sin kx$$

and showing

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{\sin \frac{1}{2}(2n+1)(t-x)}{\sin \frac{1}{2}(t-x)} dx.$$

(Is there a hint of Vieta here?)

In his proof, he assumes a finite number of discontinuities (Cauchy sense). He obtains convergence to the midpoint of jumps. He needed the continuity to gain the existence of the integral. His proof requires a *monotonicity* of f .

15. He believed his proof would adapt to an infinite number of discontinuities; which in modern terms would be *no where dense*. He promised the proof but it never came. Had he thought of extending

³Note. We have tacitly changed the interval to $[-\pi, \pi]$ for convenience.

Cauchy's integral as Riemann would do, his monotonicity condition would suffice.

16. In 1864 **Rudolf Lipschitz** (1831-1904) attempted to extend Dirichlet's analysis. He noted that an expanded notion of integral was needed. He also believed that the nowhere dense set had only a finite set of limit points. (There was no set theory at this time.) He replaced the monotonicity condition with piecewise monotonicity and what is now called a **Lipschitz condition**.

Recall, a function $f(x)$, defined on some interval $[a, b]$ is said to satisfy a Lipschitz condition of order α if for every x and y in $[a, b]$

$$|f(x) - f(y)| < c|x - y|^\alpha,$$

for some fixed constant c . Of course, Lipschitz was considering $\alpha = 1$.

Every function with a bounded derivative on an interval, J , satisfies a Lipschitz condition of order 1 on that interval. Simply take

$$c = \sup_{x \in I} |f'(x)|.$$

17. In fact Dirichlet's analysis carries over to the case when $D^{(2)} = (D')'$ is finite ($D = \text{set}$, $D' := \text{limit points of } D$), and by induction to $D^{(n)} = (D^{(n-1)})'$. (Such sets were introduced by **George Cantor** (1845-1918) in 1872.)

Example. Consider the set $D = \{1/n\}$, $n = 1, 2, \dots$. Then $D' = \{0\}$, $D^{(2)} = \emptyset$.

Example. Consider the set Z_R , of all rationals. Then $D' = R$, $D^{(2)} = R, \dots$, where R is the set of reals.

Example. Define

$$D = \left\{ \sum_{j=1}^k \frac{1}{p_j^{m_j}} \mid j = 1, 2, \dots, k, \text{ and } m_j = 1, 2, \dots \right\}$$

where the p_j are distinct primes. Then $D^{(k)} = \{0\}$.

Dirichlet may have thought for his set of discontinuities $D^{(n)}$ is finite for some n . From Dirichlet we have the beginnings of the distinction between continuous function and integrable function.

18. Dirichlet introduced the **salt-pepper** function in 1829 as an example of a function defined neither by equation nor drawn curve.

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational.} \end{cases}$$

Note. Riemann's integral cannot handle this function. To integrate this function we require the Lebesgue integral.

By way of background, another question was raging during the 19th century, that of continuity vs. differentiability. As late as 1806, the great mathematician **A-M Ampere** (1775-1836) tried without success to establish the differentiability of an arbitrary function except at "particular and isolated" values of the variable.

In fact, progress on this front did not advance during the most of the century until in 1875 **P. DuBois-Reymond** (1831-1889) gave the first conterexample of a continuous function without a derivative.

2 The Riemann Integral

Bernhard Riemann (1826-66) no doubt acquired his interest in problems connected with trigonometric series through contact with Dirichlet when he spent a year in Berlin. He almost certainly attended Dirichlet's lectures.

For his *Habilitationsschrift* (1854) Riemann under-took to study the representation of functions by trigonometric functions.

He concluded that continuous functions are represented by Fourier series. He also concluded that functions not covered by Dirichlet do not exist in nature. But there were new applications of trigonometric series to number theory and other places in pure mathematics. This provided impetus to pursue these foundational questions.

Riemann began with the question: when is a function integrable? By that he meant, **when do the Cauchy sums converge?**

He assumed this to be the case if and only if

$$(R_1) \quad \lim_{\|P\| \rightarrow 0} (D_1\delta_1 + D_2\delta_2 + \cdots + D_n\delta_n) = 0$$

where P is a partition of $[a, b]$ with δ_i the lengths of the subintervals and the D_i are the corresponding oscillations of $f(x)$:

$$D_I = \left| \sup_{x \in I} f(x) - \inf_{x \in I} f(x) \right|.$$

For a given partition P and $\delta > 0$, define

$$S = s(P, \delta) = \sum_{D_i > \delta} \delta_i.$$

Riemann proved that the following is a necessary and sufficient condition for integrability (R2):

Corresponding to every pair of positive numbers ε and σ there is a positive d such that if P is any partition with norm $\|P\| \leq d$, then $S(P, \sigma) < \varepsilon$.

These conditions (R_1) and (R_2) are germs of the idea of Jordan measurability and outer content. But the time was not yet ready for measure theory.

Thus, with (R_1) and (R_2) Riemann has integrability without explicit continuity conditions. Yet it can be proved that R -integrability implies $f(x)$ is continuous *almost everywhere*.

Riemann gives this example: Define $m(x)$ to be the integer that minimizes $|x - m(x)|$. Let

$$(x) = \begin{cases} x - m(x) & x \neq n/2, n \text{ odd} \\ 0 & x = n/2, n \text{ odd} \end{cases}$$

(x) is discontinuous at $x = n/2$ when n is odd. Now define

$$f(x) = (x) + \frac{(2x)}{2^2} + \cdots + \frac{(nx)}{n^2} + \cdots .$$

This series converges and $f(x)$ is discontinuous at every point of the form $x = m/2n$, where $(m, n) = 1$.⁴ This is a dense set. At such points the left and right limiting values of this function are

$$f(x\pm) = f(x) \mp (\pi^2/16n^2).$$

This function satisfies (R_2) and thus f is R -integrable.

The R -integral lacks important properties for limits of sequences and series of functions. The basic theorem for the limit of integrals is:

Theorem. *Let J be a closed interval $[a, b]$, and let $\{f_n(x)\}$ be a sequence of functions such that*

$$\lim_{n \rightarrow \infty} (R) \int_a^b f_n(x) dx \text{ exists}$$

and such that $f_n(x)$ tends uniformly to $f(x)$ in J as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} (R) \int_a^b f_n(x) dx = (R) \int_a^b f(x) dx.$$

That this is unsatisfactory is easily seen from an example. Consider the sequence of functions defined on $[0, 1]$ by $f_n(x) = x^n$, $n = 1, 2, \dots$. Clearly, as $n \rightarrow \infty$, $f_n(x) \rightarrow 0$ pointwise on $[0, 1)$ and $f_n(1) = 1$, for all n . Because the convergence is not uniform, we cannot conclude from the above theorem that

$$\lim_{n \rightarrow \infty} (R) \int_0^1 f_n(x) dx = 0,$$

which, of course, it is.

What is needed is something stronger. Specifically if $|f_n(x)| \leq g(x)$ and $\{f_n\}$, g are integrable and if $\lim f_n(x) = f(x)$ then f **may not** be R -integrable.

This is a basic flaw that was finally resolved with **Lebesgue integration**.

⁴Recall, $(m, n) = 1$ means m and n are relatively prime.

3 Postscript

The (incomplete) theory of trigonometric series, particularly the question of representability, continued to drive the progress of analysis. The most difficult question was this: what functions are Riemann integrable?⁵ To this one and the many other questions that arose we owe the foundations of set theory and transfinite induction as proposed by Georg Cantor. Cantor also sought conditions for convergence and defined the **derived** sets D^n . He happened on sets

$$D^\infty, D^{n^\infty}, \dots$$

and so on, which formed the basis of his transfinite sets. Another aspect was the development of function spaces⁶ and ultimately the functional analysis⁷ that was needed to understand them.

In a not uncommon reversal we see so much in mathematics; these spaces have played a major role in the analysis of solutions of the partial differential equations and trigonometric series that initiated their invention. Some of the most active research areas today are the direct descendants of the questions related to integrability.

I might add that these pursuits were fully in concordance with the fundamental philosophy laid down by the Pythagorean school more than two millennia ago.

⁵This question of course has been answered. The relevant theorem is this: **Theorem.** *Let J be a closed interval. The function $f(x)$ is R -integrable over J if and only if it is continuous almost everywhere- J .* In the case that f is non negative, these conditions in turn are equivalent to the graph of $f(x)$ being (Jordan) measurable.

⁶To name just a few, there are the Lebesgue, Hardy, Lipschitz, Sobolev, Orlicz, Lorentz and Besov spaces. Each space plays its own unique and important role in some slightly different areas of analysis.

⁷And this is an entire area of mathematics in and of itself.

4 The Mathematicians

Leonhard Euler (1707 - 1783)

was born in Basel Switzerland, the son of a Lutheran minister. Euler's father wanted his son to follow him into the church. Euler obtained his father's consent to change to mathematics after Johann Bernoulli had used his persuasion. Johann Bernoulli became his teacher. He joined the St. Petersburg Academy of Science in 1727, two years after it was founded by Catherine I He married and had 13 children altogether of which 5 survived their infancy.



He claimed that he made some of his greatest discoveries while holding a baby on his arm.

Euler is widely considered to be among a handful of the best mathematicians of all time. His contributions to almost every area of mathematics are pathfinding. In particular, his contributions to analysis and number theory remain of use even today.

In 1741, at the invitation of Frederick the Great, Euler joined the Berlin Academy of Science, where he remained for 25 years. During his time in Berlin, he wrote over 200 articles.

In 1766 Euler returned to Russia. Euler lost the sight of his right eye at the age of 31 and soon after his return to St Petersburg he became almost entirely blind after a cataract operation. Because of his remarkable memory was able to continue with his work on optics, algebra, and lunar motion. Amazingly after 1765 (when Euler was 58) he produced almost half his works despite being totally blind.

After his death in 1783 the St. Petersburg Academy continued to publish Euler's unpublished work for nearly 50 more years.

Euler made large bounds in modern analytic geometry and trigonometry. He made decisive and formative contributions to geometry, calculus and number theory. In number theory he did much work in correspondence with Goldbach. He integrated Leibniz's differential calculus and Newton's method of fluxions. He was the most prolific writer of mathematics of all time. His complete works contains 886 books and papers.

We owe to him the notations $f(x)$ (1734), e for the base of natural logs (1727), i for the square root of -1 (1777), π for pi, Σ for summation (1755) etc. He also introduced beta and gamma functions, integrating factors for differential equations.

Although Destouches never disclosed his identity as father of the child, he left his son an annuity of 1,200 livres. D'Alembert's teachers at first hoped to train him for theology, being perhaps encouraged by a commentary he wrote on St. Paul's Letter to the Romans, but they inspired in him only a lifelong aversion to the subject. He spent two years studying law and became an advocate in 1738, although he never practiced. After taking up medicine for a year, Apart from some private lessons, d'Alembert was almost entirely self-taught.

Jean Le Rond d'Alembert (1717 - 1783) D'Alembert grew up in Paris, the illegitimate son of a famous hostess, Mme de Tencin, and one of her lovers, the chevalier Destouches-Canon. He was abandoned on the steps of the Parisian church of Saint-Jean-le-Rond, whence his name. His father provided for him — as a distance, and he had the opportunity to obtain a good education. His teachers attempted to direct him toward theology, but after some attempts at medicine and law, he finally dedicated himself to mathematics — “the only occu-



pation which really interested me,” he said later in life. In mathematics he was almost entirely self-taught.

Jean d’Alembert was a pioneer in the study of differential equations and pioneered their use of in physics. He studied the equilibrium and motion of fluids.

In 1739 he read his first paper to the French Academy of Sciences, of which he became a member in 1741. At the age of 26, in 1743, he published his important *Traité de dynamique*, an important treatise on dynamics. Containing what is now known as “d’Alembert’s principle,” which states that Newton’s third law of motion (for every action there is an equal and opposite reaction) is true for bodies that are free to move as well as for bodies rigidly fixed, it secured his reputation in mathematics. Other mathematical works pured from his pen. In 1744 he published *Traité de l’équilibre et du mouvement des fluides* which applied his principle to the theory of equilibrium and motion of fluids.

Following came his fundamental papers on the development of partial differential equations. His first paper in this area won him a prize at the Berlin Academy, to which he was elected the same year. By 1747 he had applied his theories to the problem of vibrating strings. In 1749 he found an explanation of the precession of the equinoxes.

He did important work in the foundations of analysis and in 1754 in an article entitled *Différentiel* in volume 4 of *Encyclopédie* suggested that the theory of limits be put on a firm foundation. He was one of the first to understand the importance of functions and, in this article, he defined the derivative of a function as the limit of a quotient of increments. In fact he wrote most of the mathematical articles in this 28 volume work. From 1761 to 1780 he published eight volumes of his *Opuscules mathématiques*.

D’Alembert also studied hydrodynamics, the mechanics of rigid bodies, the three-body problem in astronomy and atmospheric circulation.

He was a friend of Voltaire.

He investigated not only mathematics but also Bernoulli’s theorem, which he derived, is named after him.

Daniel Bernoulli (1700 - 1782)

was the second son of Johann Bernoulli and the nephew of Jacob Bernoulli. He was clearly the most distinguished of the second generation of this famous family of scientists and mathematicians. His most important work considered the basic properties of fluid flow, pressure, density and velocity, and gave their fundamental relationship now known as Bernoulli's principle. He also studied such fields as medicine, biology, physiology, mechanics, physics, astronomy, and oceanography. In 1725 Daniel and his brother Nikolaus were invited to work at the St. Petersburg Academy of Sciences⁸. There he collaborated with Euler, who came to St. Petersburg in 1727.



In 1731 Daniel extended his researches to cover problems of life insurance.

In 1733 Daniel returned to Basel where he taught anatomy, botany, physiology and physics. His most important work was *Hydrodynamica* which considered the basic properties of fluid flow, pressure, density and velocity, and gave their fundamental relationship now known as Bernoulli's principle. He also established the basis for the kinetic theory of gases.

Between 1725 and 1749 he won ten prizes for work on astronomy, gravity, tides, magnetism, ocean currents, and the behaviour of ships at sea.

⁸Together they collaborated on what has been called the Petersburg Paradox. It goes like this: Suppose that Peter and Paul agree to play a game based on the toss of a coin. If a head is thrown on the first toss, Paul will give Peter one crown; if the first toss is a tail, but a head occurs on the second toss, Paul will give Peter two crowns; if a head appears for the first time on the third toss, Paul will give Peter four crowns; and so on, the amount to be paid if head appears for the first time on the n th toss. What should Peter pay Paul for the privilege of playing the game? The mathematical expectation seems to be finite, but in simulations the pay out is very modest. This problem was the rage in the 18th century, with many solutions being offered.

Jean Baptiste Joseph Fourier

(1768 - 1830) trained for the priesthood but did not take his vows. Instead took up mathematics studying (1794) and later teaching mathematics at the new École Normale. In 1798 he joined Napoleon's army in its invasion of Egypt as scientific advisor. He helped establish educational facilities in Egypt and carried out archaeological explorations.



He published *Théorie analytique de la chaleur* in 1822

devoted to the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions. Fourier also appears to have been the first to study linear inequalities systematically. While not producing any deep results, he observed their importance to mechanics and probability theory. Moreover, he was interested in finding the *least maximum deviation* fit to a system of linear equations. He suggested a solution by a vertex-to-vertex descent to a minimum, which is the principle behind the simplex method used today. For more information, see George B. Dantzig *Linear Programming and Extensions*, The Rand-Princeton U. Press, 1963.

Joseph Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Fourier's work provided the impetus for later work on trigonometric series and the theory of functions of a real variable.

Joseph-Louis Lagrange (1736 - 1813) was born in Turin and died in Paris. He served as professor of geometry at the Royal Artillery School in Turin from 1755 to 1766 and helped to found the Royal Academy of Science there in 1757. When Euler left the Berlin Academy of Science, Lagrange succeeded him as director of mathematics 1766. In 1787 he left Berlin to become a member of the Paris Academy of Science, where he remained for the rest of his career.



Lagrange excelled in all fields of analysis and number theory and analytical and celestial mechanics.

Lagrange survived the French Revolution while others did not. Lagrange said on the death of the chemist Lavoisier, “It took only a moment to cause this head to fall and a hundred years will not suffice to produce its like.”

During the 1790s he worked on the metric system and advocated a decimal base. He also taught at the École Polytechnique, which he helped to found. Napoleon named him to the Legion of Honour and Count of the Empire in 1808.

In 1788 he published *Mécanique analytique*, which summarised all the work done in the field of mechanics since the time of Newton and is notable for its use of the theory of differential equations. In it he transformed mechanics into a branch of mathematical analysis.

His early work on the theory of equations was to lead Galois to the idea of a group of permutations.

Niels Henrik Abel (1802 - 1829) was born in Norway, of poor means. Abel's life was dominated by poverty. However, Abel's teacher Holmboe, recognising his talent for mathematics, raised money from his colleagues to enable Abel to attend Christiania University. Abel entered the university in 1821. Soon thereafter, he won a scholarship to visit Germany and France. Abel published in 1823 papers on functional equations and integrals. In it Abel gives the first solution of an integral equation.



In 1824 he proved the impossibility of solving algebraically the general equation of the fifth degree and published it at his own expense hoping to obtain recognition for his work.

Abel was instrumental in establishing mathematical analysis on a rigorous basis. His major work *Recherches sur les fonctions elliptiques* was published in 1827 in the first volume of Crelle's Journal, the first journal devoted entirely to mathematics.

After visiting Paris he returned to Norway heavily in debt. On returning to Norway, Abel travelled by sled to visit his fiancée for Christmas 1828 in Froland. He became seriously ill (from tuberculosis) on the sled journey and died a couple of months later. ect

Augustin Louis Cauchy (1789 - 1857) was born in Paris. He pioneered the study of analysis and the theory of substitution groups (now called permutation groups). Cauchy proved in 1811 that the angles of a convex polyhedron are determined by its faces. In 1814 he published the memoir on definite integrals that became the basis of the theory of complex functions. His other contributions include researches in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.



Augustin-Louis Cauchy pioneered the study of analysis and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics. He is often called the “father of modern analysis.”

Numerous terms in mathematics bear his name: the Cauchy integral theorem, the Cauchy-Kovalevskaya existence theorem, the Cauchy integral formula, the Cauchy-Riemann equations and Cauchy sequences.

Cauchy was the first to make a rigorous study of the conditions for convergence of infinite series and he also gave a rigorous definition of an integral. His influential text *Cours d'analyse* in 1821 was designed for students and was concerned with developing the basic theorems of the calculus as rigorously as possible.

He produced 789 mathematics papers but was disliked by most of his colleagues. He displayed self-righteous obstinacy and an aggressive

religious bigotry. An ardent royalist he spent some time in Italy after refusing to take an oath of allegiance.

Henri Léon Lebesgue (1875 - 1941) studied at École Normale Supérieure. He taught in the Lycée at Nancy from 1899 to 1902. Building on the work of others, including that of the French mathematicians Emile Borel and Camille Jordan, Lebesgue formulated the theory of measure in 1901 and the following year he gave the definition of the Lebesgue integral that generalises the notion of the Lebesgue Riemann integral by extending



the concept of the area below a curve to include a sufficiently rich class of functions that the limit theorems needed in applications are simple consequences.

This particular achievement of modern analysis, which greatly expanded the scope of Fourier analysis, appears in Lebesgue's dissertation, *Intégrale, longueur, aire*, presented to the University of Nancy in 1902.

In addition to about 50 papers he wrote two major books *Leçons sur l'intégration et la recherche des fonctions primitives* (1904) and *Leçons sur les séries trigonométriques* (1906). He also made major contributions in other areas of mathematics, including topology, potential theory, and Fourier analysis.

Rudolf Otto Sigismund Lipschitz (1832 - 1903) worked on quadratic differential forms and mechanics. His work on the Hamilton-Jacobi method for integrating the equations of motion of a general dynamical system led to important applications in celestial mechanics. Lipschitz is remembered for the 'Lipschitz condition', an inequality that guarantees a unique solution to the differential equation $y' = f(x,y)$. Peano gave an existence theorem for this differential equation, giving conditions which guarantee at least one solution.



Georg Ferdinand Ludwig Philipp Cantor (1845 - 1918) was born in Russia (St. Petersburg) but lived almost his entire life in Germany. Cantor attended the University of Zürich for a term in 1862 but then went to the University of Berlin where he attended lectures by Weierstrass, Kummer and Kronecker. He received his doctorate in 1867 from Berlin and accepted a position at the University of Halle in 1869, where he remained until he retired in 1913.



Georg Cantor founded set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers. He also advanced the study of trigonometric series.

Cantor founded set theory and introduced the mathematically meaningful concept of infinite numbers with his discovery of transfinite numbers. He also advanced the study of trigonometric series and was the first to prove the nondenumerability of the real numbers.

His first papers (1870-1872) showed the influence of Weierstrass's teaching, dealing with trigonometric series. In 1872 he defined irrational numbers in terms of convergent sequences of rational numbers. In 1873 he proved the rational numbers countable, i.e. they may be placed in 1-1 correspondence with the natural numbers.

A transcendental number is an irrational number that is not a root of any polynomial equation with integer coefficients. Liouville established in 1851 that transcendental numbers exist. Twenty years later Cantor showed that in a certain sense 'almost all' numbers are transcendental.

Closely related to Cantor's work in transfinite set theory was his definition of the continuum. Cantor's work was attacked by many mathematicians, the attack being led by Cantor's own teacher Kronecker.⁹ Cantor never doubted the absolute truth of his work despite the discovery of the paradoxes of set theory. He was supported by Dedekind, Weierstrass and Hilbert, Russell and Zermelo. Hilbert described Cantor's work as "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity."

Cantor died in a psychiatric clinic in Halle in 1918.

⁹Leopold Kronecker was a particularly traditional mathematician. He is attributed to have said in 1886 at the Berlin meeting of the "Vereinigung deutscher Naturforscher und Arzte" that "Die natürlichen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk." (The natural numbers were made by God, all the rest is man made.) As such it is quoted by Heinrich Weber on p.19 of his memorial article *Leopold Kronecker*, Jahresber. DMV 2 (1892) 25-31. There is, however, no direct quotation, or anything related, in Kronecker's published works. As an advocate of constructive mathematics, he had difficulties accepting infinities, countable or uncountable.

Georg Friedrich Bernhard Riemann (1826 - 1866) moved from Göttingen to Berlin in 1846 to study under Jacobi, Dirichlet and Eisenstein. In 1849 he returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851. Riemann, on Gauss's recommendation, was appointed to a post in Göttingen. Riemann's paper *Über die Hypothesen welche der Geometrie zu Grunde liegen*, written in 1854, became a classic of mathematics, and its results were incorporated into Albert Einstein's relativistic theory of gravitation.



Bernhard Riemann's ideas concerning geometry of space have had a profound effect on the development of modern analysis that is even still being explored.

Gauss's chair at Göttingen was filled by Dirichlet in 1855 and, after his death, by Riemann.

Riemann's ideas concerning geometry of space had a profound effect on the development of modern theoretical physics and provided the concepts and methods used later in relativity theory. He was an original thinker and a host of methods, theorems and concepts are named after him.

The Cauchy-Riemann equations (known before his time) and the concept of a Riemann surface appear in his doctoral thesis. He clarified the notion of integral by defining what we now call the Riemann integral. He is also famed for the still unsolved Riemann hypothesis.

He died from tuberculosis.