ADVANCED CALCULUS

1996–1997

Gilbert Weinstein

Office:	CH 493B
Tel:	(205) 934-3724
	(205) 934-2154
FAX:	(205) 934-9025
Email:	weinstei@math.uab.edu
Office hours:	Monday 1:00 pm – 2:30 pm
	Wednesday 8:30 am – 10:00 am

About the Course

Welcome to Advanced Calculus! In this course, you will learn Analysis, that is the theory of differentiation and integration of functions. However, you will also learn something more fundamental than Analysis. You will learn what is a *mathematical proof.* You may have seen some proofs earlier, but here, you will learn how to write your own proofs. You will also learn how to understand someone else's proof, find a flaw in a proof, fix a deficient proof if possible and discard it if not. In other words you will learn the trade of mathematical exploration.

Mathematical reasoning takes time. In Calculus, you expected to read a problem and immediately know how to proceed. Here you may expect some frustration and you should plan to spend a lot of time thinking before you write down anything. Analysis was not discovered overnight. It took centuries for the correct approach to emerge. You will have to go through an accelerated process of rediscovery. Twenty hours of work a week outside class is not an unusual average for this course.

The course is run in the following way. In these notes, you will find *Definitions*, *Theorems*, and *Examples*. I will explain the definitions. At home, **on your own**, you will try to prove the theorems and the statements in the examples. You will use no books and no help from anyone. It will be just you, the pencil and the paper. Every statement you make must be justified. In your arguments, you may use any result which precedes in the notes the item you are proving. You may use these results even if they have not yet been proven. However, you may not use results from other sources. Then, in class, I will call for volunteers to present their solutions at the board. Every correct proof is worth one point. If more than one person volunteer for an item, the one with the fewest points is called to the board, ties to be broken by lot. Your grade will be determined by the number of points you have accumulated during the term.

You have to understand the proofs presented by others. Some of the ideas may be useful to you later. You must question your peers when you think a faulty argument is given or something is not entirely clear to you. If you don't, I most probably will. When you are presenting, you must make sure your arguments are clear to everyone in the class. If your proof is faulty, or you are unable to defend it, the item will go to the next volunteer, you will receive no credit, and you may not go up to the board again that day. We will work on the honor system, where you will follow the rules of the game, and I will not check on you.

CHAPTER 1

Introduction

1. Mathematical Proof

What is a proof? To explain, let us consider an example.

THEOREM 1.1. *There is no rational number r which is a square root of* 2.

This theorem was already known to the ancient Greeks. It was very important to them since they were particularly interested in geometry, and, as follows from the Theorem of Pythagoras, a segment of length $\sqrt{2}$ can be constructed as the hypotenuse of a right triangle with both sides of length 1.

Before we prove this theorem, in fact before we prove any theorem, we must understand its statement. To understand its statement, we must understand each of the terms used. For instance: what is a *rational number*? For this we need a *definition*.

DEFINITION 1.1. A number r is *rational* if it can be represented as the ratio of two integers:

$$r = \frac{n}{m} \tag{1.1}$$

where $m \neq 0$.

Of course, in this definition, we are using other terms that need to be defined, such as *number*, *ratio*, *integer*. We will not dwell on this point, and instead assume for now that these have been defined previously. However, already one point is clear. If we wish to be absolutely rigorous, we must begin from some given assumptions. We will call these *axioms*. They do not require proof. We will discuss this point further later. For the time being, let us assume that we have a system of numbers where the usual operations of arithmetic are defined.

Next we need to define what we mean by a *square root* of 2.

DEFINITION 1.2. Let y be a number. The number x is a *square root* of y if $x^2 = y$.

Again, we assume that the meaning of x^2 is understood. Note that we have said *a* square root, and not *the* square root. Indeed if $x \neq 0$ is a square root of y, then $-x$ is another one. Note that then, one of the two numbers x and $-x$ is positive. Now, we may give the proof of Theorem 1.1.

PROOF. The proof is by contradiction. Suppose that x is a square root of 2, and that x is rational. Clearly, $x \neq 0$, hence we may assume that $x > 0$. Then, $x^2 = 2$, and there are integers n, $m \neq 0$, such that

$$x = \frac{n}{m}. \tag{1.2}$$

Of course there are many such pairs n, and m. In fact, if n and m is any such pair, then $2n$ and $2m$ is another pair. Also, there is one pair in which $n > 0$. Among all these pairs, with $n > 0$, pick one for which n is the smallest positive integer possible, i.e., $x = n/m$, $n > 0$, and if $x = k/l$ then $n \leqslant k$. We have:

$$\left(\frac{n}{m}\right)^2 = 2, \tag{1.3}$$

or equivalently

$$n^2 = 2m^2. \tag{1.4}$$

Thus, 2 divides $n^2 = n \cdot n$. It follows that 2 divides n, i.e. n is even. We may therefore write $n = 2k$ and thus

$$n^2 = 4k^2 = 2m^2, \tag{1.5}$$

or equivalently

$$2k^2 = m^2. \tag{1.6}$$

Now, 2 divides m^2, hence m is even. Write $m = 2l$. We obtain

$$x = \frac{n}{m} = \frac{2k}{2l} = \frac{k}{l}. \tag{1.7}$$

But k is positive and clearly $k < n$, a contradiction. Thus no such x exists, and the theorem is proved. □

A close examination of this proof will be instructive. The first observation is that the proof is by contradiction. We assumed that the statement to be proved is false, and we reached an absurdity. Here the statement to be proved was that *there is no rational x for which* $x^2 = 2$, so we assumed there is one such x. The absurdity was that we could certainly take $x = n/m$, with $n > 0$ and as small as possible, but we deduced $x = k/l$ with $0 < k < n$. Next, we see that each step follows from the previous one, and possibly some additional information. Take for example, the argument immediately following (1.4). If 2 divides n^2 then 2 divides n. This seemingly obvious fact requires justification. We will not do this here; it is done in algebra, and relies on the unique factorization by primes of the integers. It is extremely important to identify the information which you import into your proof from outside. Usually, this is done by quoting a known theorem. Remember that before you quote a theorem, you must check its hypotheses.

Read this proof again and again during the term. Try to find its weak points, those points which could use more justification. Try to improve it. Try to imagine how it was discovered.

2. Set Theory and Notation

In this section, we briefly recall some notation and a few facts from set theory. If A is a set of objects and x is an element of A, we will write $x \in A$. If B is another set and every element of B is an element of A, we say that B is a *subset* of A, and we write $B \subset A$. In other words, $B \subset A$ if and only if $x \in B$ implies that $x \in A$. This is how one usually checks if $B \subset A$, i.e., pick an *arbitrary* element $x \in B$ and show that $x \in A$. The meaning of arbitrary here is simply that the only fact we know about x is that $x \in B$. Note also that we have used the words *if and only if*, which mean that the two statements are equivalent. We will abbreviate *if and only if* by *iff*. For example, if A and B are two sets, then $A = B$ iff $A \subset B$ and $B \subset A$.

This is usually the way one checks that two sets A and B are equal: $A \subset B$ and $B \subset A$. Again, we are learning early an important lesson: *break a proof into smaller parts.* In these notes, you will find that I have tried to break the development of the material into the proof of a great many small facts. However, in the more difficult problems, you might want to continue this process further on your own, i.e., decompose the harder problems into a number of smaller problems. Try to take the proof of Theorem 1.1 and break it into the proof of several facts. Think about how you could have guessed that these were intermediate steps in proving Theorem 1.1.

We will also use the symbol $\forall$ to mean *for every*, and the symbol $\exists$ to mean *there is*. Finally we will use $\emptyset$ to denote the *empty set*, the set with no elements,

Let A and B be sets, their *intersection*, which is denoted by $A \cap B$, is the set of all elements which belong to both A and B. Thus, $x \in A \cap B$ iff $x \in A$ and $x \in B$. If $A \cap B = \emptyset$ we say that A and B are *disjoint*. Similarly, their *union*, denoted $A \cup B$, is the set of all elements which belong to either A or B (or both), so that $x \in A \cup B$ iff $x \in A$ or $x \in B$.

Theorem 1.2. *Let A, B, and C be sets. Then*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C). \tag{1.8}$$

Theorem 1.3. *Let A, B, and C be sets. Then*

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C). \tag{1.9}$$

If A is a subset of X, then the *complement* of A in X is the set $X \setminus A$ which consists of all the elements of X which do not belong to A, i.e. $x \in X \setminus A$ iff $x \in X$ and $x \notin A$. If X is understood, then the complement is also denoted A^c. The notation $X \setminus A$ is occasionally used also when A is not a subset of X. However, in this case $X \setminus A$ is not called the complement of A in X.

Let X and Y be sets. The set of all pairs (x, y) with $x \in X$ and $y \in Y$ is called the *Cartesian product* of X and Y, and is written $X \times Y$. A subset f of $X \times Y$ is a *function*, if for each $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. The last statement includes two conditions, *existence* and *uniqueness*. These are often treated separately. Thus $f \subset X \times Y$ is a function iff:

(i) $\forall x \in X$, $\exists y \in Y$, such that $(x, y) \in f$;
(ii) If $(x, y_1) \in f$, and $(x, y_2) \in f$, then $y_1 = y_2$.

To simplify the notation, we will write $y = f(x)$ instead of $(x, y) \in f$. If $f \subset X \times Y$ is a function, we write $f\colon X \to Y$. We call X the *domain* of f, and say that f *maps X to Y*. The set of $y \in Y$ such that $\exists x \in X$ for which $y = f(x)$ is called the *range* of f, and will be written as $\mathrm{Ran}(f)$.

Let $f\colon X \to Y$. If $A \subset X$, we define the function $f|A\colon A \to Y$, called the *restriction* of f to A by setting $(f|A)(x) = f(x)$ for every $x \in A$. We define the set $f(A) \subset Y$ by

$$\begin{aligned} f(A) &= \{y \in Y\colon\ \exists x \in A, f(x) = y\} \\ &= \{f(x)\colon\ x \in A\}. \end{aligned} \tag{1.10}$$

In other words $f(A) = \mathrm{Ran}(f|A)$.

THEOREM 1.4. *Let $f\colon X \to Y$, and $A, B \subset X$, then*

$$f(A \cup B) = f(A) \cup f(B). \tag{1.11}$$

THEOREM 1.5. *Let $f\colon X \to Y$, and $A, B \subset X$, then*

$$f(A \cap B) \subset f(A) \cap f(B). \tag{1.12}$$

EXAMPLE 1.1. Equality does not always hold in (1.12).

If $A \subset Y$, we define the *pre-image of A under f*, denoted $f^{-1}(A) \subset X$, by:

$$f^{-1}(A) = \{x \in X\colon\ f(x) \in A\}. \tag{1.13}$$

THEOREM 1.6. *Let $f\colon X \to Y$, and $A, B \subset Y$, then*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B). \tag{1.14}$$

THEOREM 1.7. *Let $f\colon X \to Y$, and $A, B \subset Y$, then*

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B). \tag{1.15}$$

A function $f\colon X \to Y$ is *onto* (or *surjective*) if $\mathrm{Ran}(f) = Y$. A function $f\colon X \to Y$ is *one-to-one* (or *injective*) if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. If f is both one-to-one and onto, we say that f is *bijective*. Then there exists a unique function g such that $g(f(x)) = x$ for all $x \in X$, and $f(g(y)) = y$ for all $y \in Y$. The function g is denoted f^{-1}, and is called the *inverse* of f.

EXAMPLE 1.2. Let $f\colon X \to Y$ be bijective, and let f^{-1} be its inverse. Then $\forall y \in Y$:

$$f^{-1}(\{y\}) = \{f^{-1}(y)\} \tag{1.16}$$

Caution: part of the problem here is to explain the notation. In particular, the notation is abused in the sense that f^{-1} on the left-hand side has a different meaning than on the right-hand side.

3. Induction

Induction is an essential tool if we wish to write rigorous proof using repetitions of an argument an unknown number of times. In a more elementary course, a combination of words such as '*and so on*' could probably be used. Here, this will not be acceptable. We will denote by $\mathbb{N}$ the set of natural numbers, i.e. the counting numbers $1, 2, 3, \ldots$, and by $\mathbb{Z}$ the integers, i.e., the natural numbers, their negatives, and zero.

AXIOM 1 (Well-Ordering Axiom). Let $\emptyset \neq S \subset \mathbb{N}$. Then S contains a smallest element.

As the name indicates, we will take this as an axiom. No proof need be given. However, we will prove the *Principle of Mathematical Induction*.

THEOREM 1.8 (The Principle of Mathematical Induction). *Suppose that $S \subset \mathbb{N}$ satisfies the following two conditions:*

(i) $1 \in S$.
(ii) *If $n \in S$ then $n + 1 \in S$.*

Then $S = \mathbb{N}$.

We will denote by $P(n)$ a statement about integers. You may want to think of $P(n)$ as a function from $\mathbb{Z}$ to $\{0, 1\}$ (0 represents false, 1 represents true.) $P(n)$ might be the statement: *n is odd*; or the statement: *the number of primes less than or equal to n is less then $n/\log(n)$*.

THEOREM 1.9. *Let $P(n)$ be a statement depending on an integer n, and let $n_0 \in \mathbb{Z}$. Suppose that*

(i) *$P(n_0)$ is true;*
(ii) *If $n \geqslant n_0$ and $P(n)$ is true, then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{Z}$ such that $n \geqslant n_0$.

This last result is what we usually refer to as Mathematical Induction. Here is an application.

EXAMPLE 1.3. Let $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n}(2k-1) = n^2 \tag{1.17}$$

See Theorem 2.13, and Equation (2.6) for a precise definition of the summation's $\sum$ notation.

4. The Real Number System

In a course on the foundations of mathematics, one would construct first the natural numbers $\mathbb{N}$, then the integers $\mathbb{Z}$, then the rational numbers $\mathbb{Q}$, and then finally the real numbers $\mathbb{R}$. However, this is not a course on the foundations of mathematics, and we will not labor on these constructions. Instead, we will assume that we are given the set of real numbers $\mathbb{R}$ with all the properties we need. These will be stated as axioms. Of course, you probably already have some intuition as to what real numbers are, and these axioms are not meant to substitute for that intuition. However, when writing your proofs, you should make sure that all your statements follow from these axioms, and their consequences proved so far only.

The set of real numbers $\mathbb{R}$ is characterized as being a *complete ordered field*. Thus, the axioms for the real numbers are divided into three sets. The first set of axioms, the field axioms, are purely algebraic. The second set of axioms are the order axioms. It is extremely important to note that $\mathbb{Q}$ satisfies the axioms in the first two sets. Thus $\mathbb{Q}$ is an ordered field. Nevertheless, if one wishes to do analysis, the rational numbers are totally inadequate. Another ordered field is given in the appendix. The last axiom required in order to study analysis is *the Least Upper Bound Axiom*. An ordered field which satisfies this last axiom is said to be complete. I suggest that at first, you try to understand, or rather recognize, the first two sets of axioms.

AXIOM 2 (Field Axioms). For each pair $x, y \in \mathbb{R}$, there is an element denoted $x+y \in \mathbb{R}$, called the *sum* of x and y, such that the following properties are satisfied:

(i) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{R}$.
(ii) $x+y = y+x$ for all $x, y \in \mathbb{R}$.
(iii) There exists an element $0 \in \mathbb{R}$, called *zero*, such that $x+0 = x$ for all $x \in \mathbb{R}$.
(iv) For each $x \in \mathbb{R}$ there is an element denoted $-x \in \mathbb{R}$ such that $x+(-x) = 0$.

Furthermore, for each pair $x, y \in \mathbb{R}$, there is an element denoted $xy \in \mathbb{R}$, and called the *product* of x and y, such that the following properties are satisfied:

(v) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{R}$.
(vi) $xy = yx$ for all $x, y \in \mathbb{R}$.
(vii) There exists a non-zero element $1 \in \mathbb{R}$ such that $1 \cdot x = x$ for all $x \in \mathbb{R}$.
(viii) For all non-zero $x \in \mathbb{R}$, there is an element denoted $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$.
(ix) For all $x, y, z \in \mathbb{R}$, $(x + y)z = xz + yz$.

Any set which satisfies the above field axioms is called a *field*. Using these, it is possible to prove that all the usual rules of arithmetic hold.

EXAMPLE 1.4. Let $X = \{x\}$ be a set containing one element x, and define the operations $+$ and $\cdot$ in the only possible way. Is X a field?

EXAMPLE 1.5. Let $X = \{0, 1\}$, and define the addition and multiplication operations according to the following tables:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Prove that X is a field.

AXIOM 3 (Ordering Axioms). There is a subset $P \subset \mathbb{R}$, called the set of *positive numbers*, such that:

(x) $0 \notin P$.
(xi) Let $0 \neq x \in \mathbb{R}$. If $x \notin P$ then $-x \in P$, and if $x \in P$ then $-x \notin P$.
(xii) If $x, y \in P$, then $xy, x + y \in P$.

A field with a subset P satisfying the ordering axioms is called an *ordered field*. Let $N = -P = \{x \in \mathbb{R}\colon\ -x \in P\}$ denote the *negative numbers*. Then (x) and (xi) simply say that $\mathbb{R} = P \cup \{0\} \cup N$, and these three sets are disjoint. We define $x < y$ to mean $y - x \in P$. Thus $P = \{x \in \mathbb{R}\colon\ 0 < x\}$. The usual rules for handling inequalities follow.

THEOREM 1.10. *Let $x, y, z \in \mathbb{R}$. If $x < y$ and $y < z$ then $x < z$.*

THEOREM 1.11. *Let $x, y, z \in \mathbb{R}$. If $x < y$ and $0 < z$ then $xz < yz$.*

We will also use $x > y$ to mean $y < x$.

THEOREM 1.12.

$$0 < 1 \tag{1.18}$$

THEOREM 1.13. *Let $x, y, z \in \mathbb{R}$, then*

(i) *If $x < y$ then $x + z < y + z$.*
(ii) *If $x > 0$ then $x^{-1} > 0$.*
(iii) *If $x, y > 0$ and $x < y$ then $y^{-1} < x^{-1}$.*

We also define $x \leqslant y$ to mean that either $x < y$ or $x = y$.

THEOREM 1.14. *If $x \leqslant y$ and $y \leqslant x$ then $x = y$.*

THEOREM 1.15. *Suppose that* $x, y \in \mathbb{R}$ *and* $x < y$*, then there is* $z \in \mathbb{R}$ *such that* $x < z < y$*.*

Note that since we will only use the axioms for an ordered field, this holds also in the rational numbers $\mathbb{Q}$.

DEFINITION 1.3. Let $S \subset \mathbb{R}$. We say that S is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $x \leqslant b$ for all $x \in S$. The number b is then called an *upper bound* for S. A number c is called a *least upper bound* for S if it has the following two properties:

(i) c is an upper bound for S;
(ii) if b is an upper bound for S, then $c \leqslant b$.

THEOREM 1.16. *If a least upper bound for* S *exists then it is unique, i.e., if* c_1 *and* c_2 *are both least upper bounds for* S *then* $c_1 = c_2$*.*

Thus we can speak of *the* least upper bound of a set S.

DEFINITION 1.4. A set S is bounded below if there exists a number $b \in \mathbb{R}$ such that $b \leqslant x$ for all $x \in S$. The number b is then called a *lower bound* for S. A number c is called a *greatest lower bound* for S if it has the following properties:

(i) c is a lower bound for S;
(ii) if b is a lower bound for S, then $b \leqslant c$.

THEOREM 1.17. S *is bounded above iff* $-S = \{x \in \mathbb{R} \colon \; -x \in S\}$ *is bounded below.*

THEOREM 1.18. c *is the least upper bound of* S *iff* $-c$ *is the greatest lower bound of* $-S$*.*

We now state the Least Upper Bound Axiom.

AXIOM 4 (Least Upper Bound Axiom). Let $\emptyset \neq S \subset \mathbb{R}$ be bounded above. Then S has a least upper bound.

We list a few consequences of this axiom.

THEOREM 1.19. *Let* $\emptyset \neq S \subset \mathbb{R}$ *be bounded below. Then* S *has a greatest lower bound.*

THEOREM 1.20. *Let* $x \in \mathbb{R}$*, then there exists an integer* $n \in \mathbb{Z}$ *such that* $n > x$*.*

EXAMPLE 1.6. Let $x \in \mathbb{R}$ satisfy

$$0 \leqslant x < \frac{1}{n}, \tag{1.19}$$

for all integers $n > 0$. Then $x = 0$.

THEOREM 1.21. *Let* $y \geqslant 0$*. Then there exists a real number* $x \geqslant 0$ *such that* $x^2 = y$*.*

In particular, in contrast with Theorem 1.1, the number 2 has a real square root.

THEOREM 1.22. *Show that the set* $\mathbb{Q}$ *of rational numbers is an ordered field, but not a complete ordered field.*

Note that if $y \geqslant 0$, and x is a square root of y, then also $-x$ is a square root of y, since

$$(-x)^2 = (-1)^2 x^2 = y. \tag{1.20}$$

Thus, if $y \geqslant 0$ it has exactly one non-negative square root x.

DEFINITION 1.5. Let $y \geqslant 0$, we define *the square root* of y to be the non-negative number $x \geqslant 0$ such that $x^2 = y$, and we denote it by $x = \sqrt{y}$.

This is somewhat in disagreement with our previous definition of square roots, but we will no longer use definition 1.2, hence no problems should arise from this.

EXAMPLE 1.7. Let $a \geqslant 0$, and $b \geqslant 0$, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

DEFINITION 1.6. Let $x \in \mathbb{R}$. We define its *absolute value* $|x|$ by:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases} \tag{1.21}$$

THEOREM 1.23. *Let $x \in \mathbb{R}$, then $\sqrt{x^2} = |x|$.*

THEOREM 1.24. *Let $x, y \in \mathbb{R}$, then $|xy| = |x||y|$.*

THEOREM 1.25. *Let $x, y \in \mathbb{R}$ then*

$$|x + y| \leqslant |x| + |y|. \tag{1.22}$$

This last inequality is called *the triangle inequality.* It is used widely.

THEOREM 1.26. *Let $x, y \in \mathbb{R}$ then*

$$|x - y| \geqslant |x| - |y|. \tag{1.23}$$

THEOREM 1.27. *Let $x, y \in \mathbb{R}$ then*

$$|x - y| \geqslant \Big||x| - |y|\Big|. \tag{1.24}$$

Appendix

In this appendix, we sketch the construction of an ordered field $\mathbb{F}$ which does not satisfy the Least Upper Bound Axiom, and in which Theorem 1.20 does not hold.

We will use the set of polynomials with real coefficients:

$$\mathbb{R}[t] = \left\{ p \colon \mathbb{R} \to \mathbb{R} \colon\ p(t) = \sum_{k=0}^{n} a_k t^k ; a_k \in \mathbb{R} \right\}. \tag{1.25}$$

Note that a function $f \colon \mathbb{R} \to \mathbb{R}$ is zero iff it assigns zero to all real $t \in \mathbb{R}$. Thus, a polynomial is zero iff all its coefficients are zero. If $p \in \mathbb{R}[t]$, and $p \neq 0$, we call the coefficient of the highest power of t the *leading coefficient*.

Let $\mathbb{F}$ be the set of rational functions with real coefficients. More precisely, let

$$\mathbb{F} = \{p/q \colon\ p, q \in \mathbb{R}[t]; q \neq 0\}, \tag{1.26}$$

A rational function can be written in many ways as the ratio of two polynomials. However, we can always arrange that all common factors have been canceled, and that in the denominator q, the leading coefficient is positive. This can be achieved by multiplying numerator and denominator by (-1) without changing the function

$r = p/q$. We define addition and multiplication in $\mathbb{F}$ as usual for functions. It is easy to see that the sum and the product of two functions in $\mathbb{F}$ is again in $\mathbb{F}$. It is also not difficult to check that $\mathbb{F}$ is a field. We will not carry out all these steps here. They are, although tedious, quite straightforward. We only note that $\mathbb{F}$ contains a copy of $\mathbb{Z}$, in fact a copy of $\mathbb{R}$, namely the constant functions.

Define $P \subset \mathbb{F}$ to be the set of non-zero rational functions p/q such that the leading coefficient of pq is positive. The set P is the set of all non-zero rational functions which can be written as a ratio p/q where the leading coefficients of both p and q are positive. Now it is clear that $0 \notin P$. Also, one can check that all the axioms for P are satisfied. Thus $\mathbb{F}$ is an ordered field.

However, Theorem 1.20 does not hold in $\mathbb{F}$. In fact, consider the function $t \in \mathbb{R}[t]$. This is the function which assigns to each real number $x \in \mathbb{R}$ the real number $x \in \mathbb{R}$. Now, if $n \in \mathbb{Z}$, then $t - n$ has leading coefficient 1, hence lies in P. Thus $n < t$. Since this is true for every $n \in \mathbb{Z}$, we have found an element $t \in \mathbb{F}$ which is larger than every integer n.

Ordered fields in which Theorem 1.20 does not hold are called *non-Archimedean ordered fields*. In such a field, there are 'infinitely large' elements. We have constructed here a non-Archimedean ordered field $\mathbb{F}$. In $\mathbb{F}$, the elements t, t^2, etc, are infinitely large. Of course the Least Upper Bound axiom does not holds in $\mathbb{F}$.

Example 1.8. Find a non-empty bounded subset $S \subset \mathbb{F}$ which does not have a least upper bound?

CHAPTER 2

Sequences

1. Limits of Sequences

Definition 2.1. A *sequence of real numbers*, is a function $x\colon \mathbb{N} \to \mathbb{R}$.

When no confusion can arise, we will usually abbreviate this simply as *a sequence*. If $x\colon \mathbb{N} \to \mathbb{R}$ is a sequence, we write, in keeping with tradition, x_n instead of $x(n)$, and we will use the notation $\{x_n\}_{n=1}^{\infty}$ for the sequence x, where $x_n = x(n) \in \mathbb{R}$. Note that there is a distinction between the sequence $\{x_n\}_{n=1}^{\infty}$ and the subset $\{x_n\colon\ n \in \mathbb{N}\} \subset \mathbb{R}$.

Definition 2.2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, and let $L \in \mathbb{R}$. We say that the sequence $\{x_n\}_{n=1}^{\infty}$ *converges to* L, if for every $\varepsilon > 0$, there is an integer $N \in \mathbb{N}$ such that for all integers $n \geqslant N$, there holds:

$$|x_n - L| < \varepsilon. \tag{2.1}$$

Theorem 2.1. *Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to L_1, and also converges to L_2. Then $L_1 = L_2$.*

Thus the number L in Definition 2.2 is unique, and we may speak of *the limit* L of the sequence, if it exists.

Example 2.1. Let $c \in \mathbb{R}$, and for each $n \in \mathbb{N}$ let $x_n = c$. Then $\{x_n\}_{n=1}^{\infty}$ converges to c.

Example 2.2. Let $x_n = 1/n$ for each $n \in \mathbb{N}$. Then $\{x_n\}_{n=1}^{\infty}$ converges to 0.

Definition 2.3. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. We say that $\{x_n\}_{n=1}^{\infty}$ *converges* if there is an $L \in \mathbb{R}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to L. If a sequence does not converge, we say that it *diverges*.

Example 2.3. Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ diverges.

Theorem 2.2. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence which converges. Then $\{x_n\colon\ n \in \mathbb{N}\} \subset \mathbb{R}$ is bounded above and below.*

In order to compute, we need theorems which allow us to perform simple arithmetic operations with limits. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences. We define their sum as the sequence $\{x_n + y_n\}_{n=1}^{\infty}$, i.e. the function $(x + y)\colon \mathbb{N} \to \mathbb{R}$ which assigns to each $n \in \mathbb{N}$ the number $x_n + y_n \in \mathbb{R}$. Similarly, we define their product as the sequence $\{x_n y_n\}_{n=1}^{\infty}$.

Theorem 2.3. *Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to L, and $\{y_n\}_{n=1}^{\infty}$ converges to M. Then $\{x_n + y_n\}_{n=1}^{\infty}$ converges to $L + M$.*

THEOREM 2.4. *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *converges to* L*, and* $\{y_n\}_{n=1}^{\infty}$ *converges to* M*. Then* $\{x_n y_n\}_{n=1}^{\infty}$ *converges to* LM*.*

THEOREM 2.5. *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *converges to* L*, and let* $c \in \mathbb{R}$*. Then* $\{cx_n\}_{n=1}^{\infty}$ *converges to* cL*.*

THEOREM 2.6. *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *converges to* L*, and* $\{y_n\}_{n=1}^{\infty}$ *converges to* M*. Then* $\{x_n - y_n\}_{n=1}^{\infty}$ *converges to* $L - M$*.*

THEOREM 2.7. *Suppose that* $x_n \neq 0$ *for each* $n \in \mathbb{N}$*, that* $\{x_n\}_{n=1}^{\infty}$ *converges to* L*, and that* $L \neq 0$*. Define* $y_n = x_n^{-1}$*. Then* $\{y_n\}_{n=1}^{\infty}$ *converges to* L^{-1}*.*

EXAMPLE 2.4. Let

$$x_n = \frac{3n^2 - 1}{n^2 + n}. \tag{2.2}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to 3.

THEOREM 2.8. *Let* $\{x_n\}_{n=1}^{\infty}$*,* $\{y_n\}_{n=1}^{\infty}$*, and* $\{z_n\}_{n=1}^{\infty}$ *be sequences, and let* $n_0 \in \mathbb{N}$*. Suppose that* $x_n \leqslant y_n \leqslant z_n$ *for all* $n \geqslant n_0$*, and that* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$*. Then* $\{y_n\}_{n=1}^{\infty}$ *converges to* L*.*

We will denote $\mathbb{N}_m = \{n \in \mathbb{Z}\colon\ n \geqslant m\}$.

THEOREM 2.9 (Recursion). *Let* $m \in \mathbb{Z}$*, let* $g\colon \mathbb{N}_m \times \mathbb{R} \to \mathbb{R}$*, and let* $a \in \mathbb{R}$*. Then, there is a unique function* $f\colon \mathbb{N}_m \to \mathbb{R}$ *such that*

(i) $f(m) = a$*;*
(ii) $f(n+1) = g(n, f(n))$*, for every* $n \in \mathbb{N}_m$*.*

THEOREM 2.10. *There exists a unique function* $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ *such that* $f(0) = 1$*, and such that* $f(n+1) = (n+1)f(n)$ *for each* $n \in \mathbb{N}_0$*.*

DEFINITION 2.4. Denote $f(n) = n!$, where $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ is the function given in Theorem 2.10. If $k, n \in \mathbb{N}_0$, we define the *binomial coefficients* by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } k \leqslant n \\ 0 & \text{if } k > n. \end{cases} \tag{2.3}$$

THEOREM 2.11. *Let* $k, n \in \mathbb{N}$*, and* $k \leqslant n$*. Then*

$$\binom{n}{k} = \binom{n}{n-k} \tag{2.4}$$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}. \tag{2.5}$$

THEOREM 2.12. *Let* $a \in \mathbb{R}$*. Then, there is a unique function* $f\colon \mathbb{N} \to \mathbb{R}$ *such that* $f(0) = 1$*, and* $f(n+1) = a\,f(n)$*.*

The function f given by Theorem 2.12 is called the *power* function, and we denote it as $f(n) = a^n$.

THEOREM 2.13 (Summation). *Let* $x\colon \mathbb{N}_m \to \mathbb{R}$*, and let* $k \geqslant m$*, then there is a unique function* $f\colon \mathbb{N}_k \to \mathbb{R}$ *such that* $f(k) = x_k$*, and* $f(n+1) = f(n) + x_{n+1}$ *for each* $n \in \mathbb{N}_k$*.*

We denote:

$$f(n) = \sum_{j=k}^{n} x_j, \tag{2.6}$$

where f is the function given in Theorem 2.13.

THEOREM 2.14. *Let $a, b \in \mathbb{R}$, and let $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{2.7}$$

EXAMPLE 2.5. Let $c \in \mathbb{R}$, and suppose $c > 0$. Define

$$x_n = \frac{1}{(1+c)^n}. \tag{2.8}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to 0.

EXAMPLE 2.6. Let $c \in \mathbb{R}$, and suppose $|c| < 1$. Define

$$x_n = c^n. \tag{2.9}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to 0.

EXAMPLE 2.7. Let $a \in \mathbb{R}$, and suppose $a \neq 1$. Then

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}. \tag{2.10}$$

EXAMPLE 2.8. Let $a \in \mathbb{R}$, and suppose $|a| < 1$. Define

$$x_n = \sum_{k=0}^{n} a^k. \tag{2.11}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $1/(1-a)$.

DEFINITION 2.5. Let $S \subset \mathbb{R}$, and let $f\colon S \to \mathbb{R}$ have the following property:

$$\text{if } a, b \in S, \text{ and } a < b, \text{ then } f(a) < f(b).$$

Then we say that f is *increasing*. Similarly, if f has the following property:

$$\text{if } a, b \in S, \text{ and } a < b, \text{ then } f(a) > f(b),$$

then we say that f is *decreasing*.

DEFINITION 2.6. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, and let $g\colon \mathbb{N} \to \mathbb{N}$ be an increasing sequence of natural numbers. Then the sequence $\{x_{g(n)}\}_{n=1}^{\infty}$ is called a *subsequence* of the sequence $\{x_n\}_{n=1}^{\infty}$.

We often write $g(j) = n_j$, and thus, we also often write the subsequence $\{x_{g(n)}\}_{n=1}^{\infty}$ as $\{x_{n_j}\}_{j=1}^{\infty}$.

THEOREM 2.15. *Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to L, and let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. Then $\{x_{n_j}\}_{j=1}^{\infty}$ converges to L.*

EXAMPLE 2.9. The converse of the previous theorem is false, i.e., there is a sequence $\{x_n\}_{n=1}^{\infty}$ which diverges, but which has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges.

When the sequence $\{x_n\}_{n=1}^{\infty}$ converges to L, we will write:

$$\lim_{n\to\infty} x_n = L. \tag{2.12}$$

2. Cauchy Sequences

Definition 2.7. A sequence $\{x_n\}_{n=1}^{\infty}$ is *non-decreasing* if $x_n \leqslant x_{n+1}$ for each $n \in \mathbb{N}$. Similarly, a sequence $\{x_n\}_{n=1}^{\infty}$ is *non-increasing* if $x_n \geqslant x_{n+1}$ for each $n \in \mathbb{N}$.

Definition 2.8. A sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded above* if the set $\{x_n\colon\ n \in \mathbb{N}\}$ is bounded above. A sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded below* if the set $\{x_n\colon\ n \in \mathbb{N}\}$ is bounded below. A sequence is *bounded* if it is bounded above and bounded below.

Theorem 2.16. *Let $\{x_n\}_{n=1}^{\infty}$ be an non-decreasing sequence which is bounded above. Let $L \in \mathbb{R}$ be the least upper bound of the set $\{x_n\colon\ n \in \mathbb{N}\}$. Then $\{x_n\}_{n=1}^{\infty}$ converges to L.*

Theorem 2.17. *Let $\{x_n\}_{n=1}^{\infty}$ be a non-increasing sequence which is bounded below. Let $L \in \mathbb{R}$ be the greatest lower bound of $\{x_n\colon\ n \in \mathbb{N}\}$. Then $\{x_n\}_{n=1}^{\infty}$ converges to L.*

Definition 2.9. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. We say that $\{x_n\}_{n=1}^{\infty}$ is a *Cauchy sequence* if the following holds:

For each $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that for all integers $n, m \geqslant N$, it holds $|x_n - x_m| < \varepsilon$.

Theorem 2.18. *Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to L. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.*

The natural question is then: Does every Cauchy sequence converge? The rest of this section is devoted to the proof of this fact.

Theorem 2.19. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence, and let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. Suppose that $\{x_{n_j}\}_{j=1}^{\infty}$ converges to L. Then $\{x_n\}_{n=1}^{\infty}$ converges to L.*

Theorem 2.20. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{x_n\}_{n=1}^{\infty}$ is bounded.*

Let $S \subset \mathbb{R}$ be bounded above. We will denote its least upper bound by $\sup S$. Let S be bounded below. We will denote its greatest lower bound by $\inf S$. Note that if $S \subset \mathbb{R}$ is bounded above, and $S' \subset S$, then S' is also bounded above. Thus if $\{x_n\}_{n=1}^{\infty}$ is bounded above, then for every $n \in \mathbb{N}$ the sets $\{x_k\colon\ k \geqslant n\}$ are bounded above.

Theorem 2.21. *Suppose $\emptyset \neq T \subset S \subset \mathbb{R}$. Then:*

$$\sup T \leqslant \sup S \tag{2.13}$$

$$\inf T \geqslant \inf S. \tag{2.14}$$

Theorem 2.22. *Suppose $\{x_n\}_{n=1}^{\infty}$ is bounded. Define*

$$y_n = \sup\{x_k\colon\ k \geqslant n\}. \tag{2.15}$$

Then the sequence $\{y_n\}_{n=1}^{\infty}$ converges.

Definition 2.10. The limit of the sequence $\{y_n\}_{n=1}^{\infty}$ defined in Theorem 2.22 is called the *limit superior* of $\{x_n\}_{n=1}^{\infty}$ and is denoted $\limsup_{n\to\infty} x_n$.

Thus, if $\{x_n\}_{n=1}^{\infty}$ is bounded,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup\{x_k\colon\ k \geqslant n\}\right). \tag{2.16}$$

Theorem 2.23. *Suppose $\{x_n\}_{n=1}^{\infty}$ is bounded. Define*

$$y_n = \inf\{x_k\colon\ k \geqslant n\}. \tag{2.17}$$

Then the sequence $\{y_n\}_{n=1}^{\infty}$ converges.

Definition 2.11. The limit of the sequence $\{y_n\}_{n=1}^{\infty}$ defined in Theorem 2.23 is called the *limit inferior* of $\{x_n\}_{n=1}^{\infty}$ and is denoted $\liminf_{n\to\infty} x_n$.

Thus, the *limit inferior* of a bounded sequence $\{x_n\}_{n=1}^{\infty}$ is:

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf\{x_k\colon\ k \geqslant n\}\right). \tag{2.18}$$

Theorem 2.24. *Let $\{x_n\}_{n=1}^{\infty}$ be bounded, and let $L = \limsup_{n\to\infty} x_n$. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges to L.*

Theorem 2.25. *If $\{x_n\}_{n=1}^{\infty}$ is bounded, it has a convergent subsequence.*

Theorem 2.26. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence, and let L be its limit superior. Then $\{x_n\}_{n=1}^{\infty}$ converges to L.*

Theorem 2.27. *A sequence $\{x_n\}_{n=1}^{\infty}$ converges iff it is a Cauchy sequence.*

Theorem 2.28. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence, and let L be its limit inferior. Then $\{x_n\}_{n=1}^{\infty}$ converges to L.*

Theorem 2.29. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Then $\{x_n\}_{n=1}^{\infty}$ converges if and only if*

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n. \tag{2.19}$$

CHAPTER 3

Series

1. Infinite Series

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We will not define what is the *series*:

$$\sum_{n=1}^{\infty} x_n. \tag{3.1}$$

This is reminiscent of axiomatic geometry where points and lines are not defined, only the relationship between them is defined. Here instead, we will define the terms *the series converges*, *the series diverges*, *the sum of the series*. Of course, one would like to think of (3.1) as an infinite sum. However, not every series has a sum, and at this point it is recommended that you try to forget everything you know about series.

THEOREM 3.1. *Let $\{x_n\}_{n=1}^{\infty}$, and $\{y_n\}_{n=1}^{\infty}$ be sequences, and let $m \in \mathbb{N}$. Then:*

$$\sum_{n=1}^{m}(x_n + y_n) = \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n. \tag{3.2}$$

THEOREM 3.2. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, let $c \in \mathbb{R}$, and let $m \in \mathbb{N}$. Then:*

$$\sum_{n=1}^{m} c x_n = c \sum_{n=1}^{m} x_n. \tag{3.3}$$

THEOREM 3.3. *Let $\{x_n\}_{n=1}^{\infty}$, and $\{y_n\}_{n=1}^{\infty}$ be sequences, and let $m \in \mathbb{N}$. Then:*

$$\left(\sum_{n=1}^{m} x_n\right)\left(\sum_{n=1}^{m} y_n\right) = \sum_{n=1}^{m}\left(\sum_{k=1}^{m} x_n y_k\right) = \sum_{k=1}^{m}\left(\sum_{n=1}^{m} x_n y_k\right). \tag{3.4}$$

THEOREM 3.4. *Let $\{x_n\}_{n=1}^{\infty}$, and $\{y_n\}_{n=1}^{\infty}$ be sequences, let $m \in \mathbb{N}$, and suppose that $x_n \leqslant y_n$ for each $n \in \mathbb{N}$ such that $n \leqslant m$. Then:*

$$\sum_{n=1}^{m} x_n \leqslant \sum_{n=1}^{m} y_n. \tag{3.5}$$

THEOREM 3.5. *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, and suppose that $x_n \leqslant y_n$ for each $n \in \mathbb{N}$ such that $n \leqslant m$. If*

$$\sum_{n=1}^{m} x_n = \sum_{n=1}^{m} y_n, \tag{3.6}$$

then $x_n = y_n$ for each $n \in \mathbb{N}$ such that $n \leqslant m$.

DEFINITION 3.1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, and consider the series (3.1). We define the *sequence of partial sums* $\{s_m\}_{m=1}^{\infty}$ of the series (3.1) by:

$$s_m = \sum_{n=1}^{m} x_n, \tag{3.7}$$

for $m \in \mathbb{N}$. We say that the series (3.1) *converges* if the sequence of partial sums $\{s_m\}_{m=1}^{\infty}$ converges. In this case, we define the *sum of the series* as the limit of the sequence $\{s_m\}_{m=1}^{\infty}$, and we write:

$$\sum_{n=1}^{\infty} x_n = \lim_{m\to\infty} s_m. \tag{3.8}$$

If the series (3.1) does not converge, we say that it *diverges*. Similarly, if $m \in \mathbb{Z}$, and $x\colon N_m \to \mathbb{R}$, then we define the partial sums of the series $\sum_{n=m}^{\infty} x_n$ by:

$$s_k = \sum_{n=m}^{k} x_n, \tag{3.9}$$

and define its sum by:

$$\sum_{n=m}^{\infty} x_n = \lim_{k\to\infty} s_k, \tag{3.10}$$

provided the series converges, i.e., provided the limit on the right-hand side of (3.10) exists.

EXAMPLE 3.1. Let $-1 < a < 1$, then

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}. \tag{3.11}$$

EXAMPLE 3.2.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \tag{3.12}$$

THEOREM 3.6. *Suppose that both series* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *converge. Then* $\sum_{n=1}^{\infty}(x_n + y_n)$ *converges, and*

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n. \tag{3.13}$$

THEOREM 3.7. *Suppose that the series* $\sum_{n=1}^{\infty} x_n$ *converges, and let* $c \in \mathbb{R}$. *Then the series* $\sum_{n=1}^{\infty} cx_n$ *converges, and*

$$\sum_{n=1}^{\infty} cx_n = c\sum_{n=1}^{\infty} x_n. \tag{3.14}$$

THEOREM 3.8. *If the series* $\sum_{n=1}^{\infty} x_n$ *converges, then* $\lim_{n\to\infty} x_n = 0$.

EXAMPLE 3.3. Show that the series $\sum_{n=1}^{\infty}(-1)^n$ does not converge.

THEOREM 3.9. *Let $m \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} x_n$ converges iff the series $\sum_{n=m+1}^{\infty} x_n$ converges. Furthermore, in this case, we have:*

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{m} x_n + \sum_{n=m+1}^{\infty} x_n \tag{3.15}$$

THEOREM 3.10. *The series $\sum_{n=1}^{\infty} x_n$ converges iff the following criterion is satisfied:*

For each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that if $m \geqslant k > N$, then:

$$\left| \sum_{n=k}^{m} x_n \right| < \varepsilon. \tag{3.16}$$

THEOREM 3.11. *Suppose that the series $\sum_{n=1}^{\infty} x_n$ converges, and let $y_m = \sum_{n=m}^{\infty} x_n$. Then $\lim_{m\to\infty} y_m = 0$.*

2. Series with Nonnegative Terms

THEOREM 3.12. *Let $N \in \mathbb{N}$ and suppose that $x_n \geqslant 0$ for all $n \geqslant N$. Then, the series $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums is bounded.*

THEOREM 3.13. *Let $N \in \mathbb{N}$, and suppose that $0 \leqslant x_n \leqslant y_n$ for all $n \geqslant N$. If the series $\sum_{n=1}^{\infty} y_n$ converges, then the series $\sum_{n=1}^{\infty} x_n$ converges, and*

$$\sum_{n=N}^{m} x_n \leqslant \sum_{n=N}^{m} y_n. \tag{3.17}$$

If the series $\sum_{n=1}^{\infty} x_n$ diverges, then the series $\sum_{n=1}^{\infty} y_n$ diverges.

EXAMPLE 3.4. Does the series $\sum_{n=1}^{\infty} 1/n^2$ converge?

THEOREM 3.14. *Suppose that $x_n > 0$ for all $n \in \mathbb{N}$, and suppose also that*

$$\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} < 1. \tag{3.18}$$

Then the series $\sum_{n=1}^{\infty} x_n$ converges.

THEOREM 3.15. *Suppose that $x_n > 0$ for all $n \in \mathbb{N}$, and suppose also that*

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} > 1. \tag{3.19}$$

Then the series $\sum_{n=1}^{\infty} x_n$ diverges.

THEOREM 3.16. *Let $x \geqslant 0$, then there is $y \geqslant 0$ such that $y^n = x$.*

We call the number y given by Theorem 3.16 *the (nonnegative) n-th root of x*, and write $y = x^{1/n}$.

THEOREM 3.17. *Let $x_n \geqslant 0$, and suppose that:*

$$\limsup_{n\to\infty} \big(x_n\big)^{1/n} < 1. \tag{3.20}$$

Then the series $\sum_{n=1}^{\infty} x_n$ converges.

THEOREM 3.18. *Let $x_n \geqslant 0$, and suppose that:*

$$\limsup_{n\to\infty}\bigl(x_n\bigr)^{1/n} > 1. \tag{3.21}$$

Then the series $\sum_{n=1}^{\infty} x_n$ diverges.

THEOREM 3.19 (Cauchy). *Suppose $\{x_n\}_{n=1}^{\infty}$ is a non-increasing sequence of nonnegative real numbers, and define*

$$y_n = 2^n x_{2^n}. \tag{3.22}$$

If the series $\sum_{n=1}^{\infty} x_n$ converges, then the series $\sum_{n=1}^{\infty} y_n$ converges.

EXAMPLE 3.5. Prove that the series $\sum_{n=1}^{\infty} 1/n$ diverges.

3. Absolute Convergence

DEFINITION 3.2. The series $\sum_{n=1}^{\infty} x_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} |x_n|$ converges. If the series $\sum_{n=1}^{\infty} x_n$ converges, but does not converge absolutely, then we say that it *converges conditionally.*

THEOREM 3.20. *If the series $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it converges.*

EXAMPLE 3.6. Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \tag{3.23}$$

converges.

THEOREM 3.21. *Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\{x_n\}_{n=1}^{\infty}$ and set $s_0 = 0$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence. Then, we have for any $k, m \in \mathbb{N}$ such that $k < m$*

$$\sum_{j=k}^{m} x_j y_j = s_m y_m - s_{k-1} y_k + \sum_{j=k}^{m-1} s_j\bigl(y_j - y_{j+1}\bigr) \tag{3.24}$$

THEOREM 3.22. *Suppose that $\{y_n\}_{n=1}^{\infty}$ is a non-increasing sequence of nonnegative real numbers which converges to 0, and suppose that the sequence of partial sums of $\{x_n\}_{n=1}^{\infty}$ is bounded. Then, the series $\sum_{n=1}^{\infty} x_n y_n$ converges.*

THEOREM 3.23 (Leibnitz). *Let $\{x_n\}_{n=1}^{\infty}$ be a non-increasing sequence of nonnegative real numbers, and suppose that $\lim_{n\to\infty} x_n = 0$. Then, then series*

$$\sum_{n=1}^{\infty} (-1)^n x_n \tag{3.25}$$

converges.

EXAMPLE 3.7. Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tag{3.26}$$

converges conditionally.

DEFINITION 3.3. Let $g\colon \mathbb{N} \to \mathbb{N}$ be one-to-one and onto. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences such that $y_n = x_{g(n)}$. Then, we say that $\sum_{n=1}^{\infty} y_n$ is a rearrangement of $\sum_{n=1}^{\infty} x_n$.

THEOREM 3.24. *Suppose that* $\sum_{n=1}^{\infty} x_n$ *converges absolutely, and let* $\sum_{n=1}^{\infty} y_n$ *be a rearrangement of* $\sum_{n=1}^{\infty} x_n$. *Then* $\sum_{n=1}^{\infty} y_n$ *converges, and*

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n. \tag{3.27}$$

EXAMPLE 3.8. Suppose the series $\sum_{n=1}^{\infty} x_n$ converges conditionally. Then there is a rearrangement $\sum_{n=1}^{\infty} y_n$ of $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} y_n$ diverges.

EXAMPLE 3.9. Suppose that the series $\sum_{n=1}^{\infty} x_n$ converges conditionally, and let $s \in \mathbb{R}$. Then, there is a rearrangement $\sum_{n=1}^{\infty} y_n$ of $\sum_{n=1}^{\infty} x_n$ such that

$$\sum_{n=1}^{\infty} y_n = s. \tag{3.28}$$

THEOREM 3.25. *Suppose that the series* $\sum_{n=0}^{\infty} x_n$ *converges absolutely, and that the series* $\sum_{n=0}^{\infty} y_n$ *converges. Define for each* $n \in \mathbb{N}$*:*

$$z_n = \sum_{k=0}^{n} x_k y_{n-k}. \tag{3.29}$$

Then, the series $\sum_{n=0}^{\infty} z_n$ *converges, and:*

$$\sum_{n=0}^{\infty} z_n = \left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{n=0}^{\infty} y_n\right). \tag{3.30}$$

CHAPTER 4

Functions and Continuity

1. Open and Closed Sets of $\mathbb{R}$

DEFINITION 4.1. Let $D \subset \mathbb{R}$, and $a \in \mathbb{R}$. We say that a is a *limit point* of D if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in D$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = a$. The set of all limit points of D is denoted $\overline{D}$.

EXAMPLE 4.1. Let $D \subset \mathbb{R}$, then $D \subset \overline{D}$. However, it may be that $D \neq \overline{D}$.

DEFINITION 4.2. A set $D \subset \mathbb{R}$ is *closed* if $\overline{D} = D$. A set $D \subset \mathbb{R}$ is *open* if $\mathbb{R} \setminus D$ is closed.

THEOREM 4.1. *Let $D \subset \mathbb{R}$. Then $\overline{D}$ is closed.*

DEFINITION 4.3. Let $a < b$, $I = \{x \in \mathbb{R}\colon\ a < x < b\}$, and $J = \{x \in \mathbb{R}\colon\ a \leqslant x \leqslant b\}$. I is called an *open interval*, and is denoted (a, b). J is called a *closed interval*, and is denoted $[a, b]$.

THEOREM 4.2. *Let $D \subset \mathbb{R}$. Then D is open if and only if the following statement holds: for every $x \in D$, there exists an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset D$.*

EXAMPLE 4.2. Every open interval is open. Every closed interval is closed.

EXAMPLE 4.3. A set may be neither open nor closed.

THEOREM 4.3. *If a set $D \subset \mathbb{R}$ is both open and closed, then either $D = \mathbb{R}$ or $D = \emptyset$.*

DEFINITION 4.4. Let $D \subset \mathbb{R}$. We define the *boundary* of D by

$$\partial D = \overline{D} \cap \overline{(\mathbb{R} \setminus D)}. \tag{4.1}$$

Let $D \subset \mathbb{R}$. We define the *interior* of D by:

$$\operatorname{int} D = D \setminus \partial D. \tag{4.2}$$

THEOREM 4.4. *Let $D \subset \mathbb{R}$, then $x \in \operatorname{int} D$ if and only if there is $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset D$.*

THEOREM 4.5. *Let $D \subset \mathbb{R}$, then D is open if and only if $D = \operatorname{int} D$.*

DEFINITION 4.5. Let $D \subset \mathbb{R}$. A set $U \subset D$ is *relatively open in D*, if there exists an open set $V \subset \mathbb{R}$ such that $V \cap D = U$.

EXAMPLE 4.4. A subset $U \subset D$ may be relatively open in D but not open. However, if D is open and U is relatively open in D, then U is open.

Let $\{U_\alpha\colon\ \alpha \in A\}$ be a family of sets. The set A is the *index* set, and it can be finite or infinite. If $A = \mathbb{N}$ for instance, we have a sequence of sets, but we may want to consider even more general situations. We define the union $\bigcup_{\alpha \in A} U_\alpha$ to be the set of all elements x which belong to at least one U_α. Thus, $x \in \bigcup_{\alpha \in A} U_\alpha$ iff $\exists \alpha \in A$ such that $x \in U_\alpha$. Similarly, we define the intersection $\bigcap_{\alpha \in A} U_\alpha$ to be the set of all elements x which belong to all the U_α. Thus, $x \in \bigcap_{\alpha \in A} U_\alpha$ iff $\forall \alpha \in A$ we have $x \in U_\alpha$.

THEOREM 4.6. *Let $\{U_j\colon\ 0 \leqslant j \leqslant n\}$ be a finite family of open subsets of $\mathbb{R}$. Then $\bigcap_{j=0}^{n} U_j$ is open.*

EXAMPLE 4.5. The intersection of a collection of open sets may not be open.

THEOREM 4.7. *Let $\{U_\alpha\colon\ \alpha \in A\}$ be a family of open subsets of $\mathbb{R}$. Then $\bigcup_{\alpha \in A} U_\alpha$ is open.*

THEOREM 4.8. *The union of a finite number of closed sets is closed. The intersection of an arbitrary collection of closed sets is closed.*

DEFINITION 4.6. A set $K \subset \mathbb{R}$ is *compact* if every sequence in K has a subsequence which converges to an element of K.

THEOREM 4.9. *Let $K \subset \mathbb{R}$ be compact. Then K is closed and bounded.*

THEOREM 4.10. *Let $K \subset \mathbb{R}$ be closed and bounded. Then K is compact.*

2. Limit and Continuity

DEFINITION 4.7. Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, $a \in \overline{D}$, and $L \in \mathbb{R}$. We say that L is the *limit of f at a* if the following condition holds:

If $x_n \in D$, and $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} f(x_n) = L$.

Remark. In other words, we require that for any sequence $\{x_n\}_{n=1}^{\infty}$ of elements in D which converges to a, the sequence $\{f(x_n)\}_{n=0}^{\infty}$ converges to L. When L is the limit of the function f at a, we write

$$\lim_{x\to a} f(x) = L. \tag{4.3}$$

THEOREM 4.11. *Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, $a \in \overline{D}$, and $L, M \in \mathbb{R}$. Suppose that both $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} f(x) = M$. Then $L = M$.*

Hence, if $f\colon D \to \mathbb{R}$ has a limit at $a \in \overline{D}$, then that limit is unique. Thus, we may speak of *the* limit of the function f at a. Next, we give an equivalent condition for L to be the limit of f at a.

EXAMPLE 4.6. Let $a, c \in \mathbb{R}$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be the function which sends $x \in \mathbb{R}$ to c. Then

$$\lim_{x\to a} f(x) = c. \tag{4.4}$$

EXAMPLE 4.7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function which sends $x \in \mathbb{R}$ to x, and let $a \in \mathbb{R}$. Then

$$\lim_{x\to a} f(x) = a. \tag{4.5}$$

THEOREM 4.12. *Let* $D \subset \mathbb{R}$, $f, g\colon D \to \mathbb{R}$, $a \in \overline{D}$, $L, M \in \mathbb{R}$. *Suppose that* $\lim_{x\to a} f(x) = L$, *and* $\lim_{x\to a} g(x) = M$.

$$\lim_{x\to a}\big(f(x)+g(x)\big) = L + M, \tag{4.6}$$

$$\lim_{x\to a}\big(f(x)g(x)\big) = LM. \tag{4.7}$$

EXAMPLE 4.8. Let $D \subset \mathbb{R}$, $a \in \overline{D}$, $c \in \mathbb{R}$, and $f\colon D \to \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = L$. Then $\lim_{x\to a} cf(x) = cL$.

THEOREM 4.13. *Let* $D \subset \mathbb{R}$, $a \in \overline{D}$, $f, g\colon D \to \mathbb{R}$, $L, M \in \mathbb{R}$. *Suppose* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, $M \neq 0$, *and that for every* $x \in D$, $g(x) \neq 0$. *Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}. \tag{4.8}$$

DEFINITION 4.8. Let $a_k \in \mathbb{R}$, for $k = 0, 1, \ldots, n$. A function $p\colon \mathbb{R} \to \mathbb{R}$ which can be written as

$$f(x) = \sum_{k=0}^{n} a_k x^k \tag{4.9}$$

is called a *polynomial*. The set of all polynomials is denoted by $\mathbb{R}[x]$. Let $p, q \in \mathbb{R}[x]$, $q \neq 0$, and let $D \subset \mathbb{R}$ be the set of all $x \in \mathbb{R}$ such that $q(x) \neq 0$. The function $r\colon D \to \mathbb{R}$, given by

$$r(x) = \frac{p(x)}{q(x)} \tag{4.10}$$

is called a *rational function*.

THEOREM 4.14. *Let* $p \in \mathbb{R}[x]$, *and let* $a \in \mathbb{R}$. *Then*

$$\lim_{x\to a} p(x) = p(a). \tag{4.11}$$

THEOREM 4.15. *Let* $r\colon D \to \mathbb{R}$ *be a rational function, and let* $a \in D$. *Then*

$$\lim_{x\to a} r(x) = r(a). \tag{4.12}$$

DEFINITION 4.9. Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, and $a \in D$. We say that f is *continuous* at a if

$$\lim_{x\to a} f(x) = f(a). \tag{4.13}$$

We say that f is continuous on D if for each $a \in D$, f is continuous at a.

EXAMPLE 4.9. Let $r\colon D \to \mathbb{R}$ be a rational function, then r is continuous on D.

THEOREM 4.16. *Let* $D \subset \mathbb{R}$, *and let* $f, g\colon D \to \mathbb{R}$. *Suppose that* f, g *are continuous at* $a \in D$. *Then* $f + g$ *and* fg *are continuous at* a.

THEOREM 4.17. *Let* $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, $a \in \overline{D}$, *and* $L \in \mathbb{R}$. *The function* f *has the limit* L *at* a *if and only if the following condition holds:*

> *For every* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if* $x \in D$ *and* $|x - a| < \delta$, *then* $|f(x) - L| < \varepsilon$.

THEOREM 4.18. *Let* $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, $a \in \overline{D}$, *and* $L \in \mathbb{R}$. *Then* f *has the limit* L *at* a *if and only if the following condition holds:*

For every open set $W \subset \mathbb{R}$, such that $L \in W$, there is an open set $U \in \mathbb{R}$ such that $a \in U$ and $f(U \cap D) \subset W$.

THEOREM 4.19. *Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$. Then f is continuous on D if and only if the following condition holds:*

For every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is relatively open in D.

THEOREM 4.20. *Let $f\colon D \to \mathbb{R}$ be a continuous function, and suppose that D is compact. Then $f(D)$ is compact.*

DEFINITION 4.10. A function $f\colon D \to \mathbb{R}$ is said to be *bounded above* if $f(D)$ is bounded above. For such a function, we will write:

$$\sup_D f = \sup f(D). \tag{4.14}$$

The function f is said to be *bounded below* if $f(D)$ is bounded below. For such a function, we will write:

$$\inf_D f = \inf f(D). \tag{4.15}$$

It is said to be *bounded* if it is bounded above and below.

EXAMPLE 4.10. There is a function $f\colon D \to \mathbb{R}$ bounded above for which there is no $x \in D$ such that $f(x) = \sup_D f$.

THEOREM 4.21. *Let D be compact, and let $f\colon D \to \mathbb{R}$ be a continuous function. Then there is $x \in D$ such that $f(x) = \sup_D f$.*

THEOREM 4.22. *Let D be compact, and let $f\colon D \to \mathbb{R}$ be a continuous function. Then there is $x \in D$ such that $f(x) = \inf_D f$.*

THEOREM 4.23. *Let $a, b \in \mathbb{R}$, $a < b$, let $f\colon [a,b] \to \mathbb{R}$ be continuous, and suppose that*

$$f(a) \leqslant 0 \leqslant f(b). \tag{4.16}$$

Then there is $x \in [a,b]$ such that $f(x) = 0$.

3. Uniform Continuity and Uniform Convergence

DEFINITION 4.11. Let $D \subset \mathbb{R}$, and let $f\colon D \to \mathbb{R}$. We say that f is *uniformly continuous* on D is for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in D$ which satisfy $|x - y| < \delta$ there holds $|f(x) - f(y)| < \varepsilon$.

THEOREM 4.24. *Let $D \subset \mathbb{R}$ be bounded, and let $f\colon D \to \mathbb{R}$ be uniformly continuous. Then f is bounded.*

EXAMPLE 4.11. Give a domain $D \subset \mathbb{R}$ and a function $f\colon D \to \mathbb{R}$ which is continuous on D but not uniformly continuous.

THEOREM 4.25. *Let D be compact, and let $f\colon D \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

DEFINITION 4.12. Let $D \subset \mathbb{R}$. A *sequence of functions* $\{f_n\}_{n=1}^{\infty}$ on D is a map from $\mathbb{N}$ into the set of functions $\mathbb{R}^D = \{f\colon D \to \mathbb{R}\}$. If for each $n \in \mathbb{N}$ the function f_n is continuous on D, we say that $\{f_n\}_{n=1}^{\infty}$ is a *sequence of continuous functions* on D. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on D, and let $f\colon D \to \mathbb{R}$. We say that $\{f_n\}_{n=1}^{\infty}$ converges to f if for each $x \in D$ the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$.

EXAMPLE 4.12. There exists $D \subset \mathbb{R}$, and a sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ on D which converges to a function $f\colon D \to \mathbb{R}$ which is not continuous.

DEFINITION 4.13. Let $D \subset \mathbb{R}$, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on D, and let $f\colon D \to \mathbb{R}$. We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to f on D if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ which satisfy $n \geqslant N$, and for all $x \in D$, there holds $|f_n(x) - f(x)| < \varepsilon$.

THEOREM 4.26. *Let $D \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on D which converges uniformly to f. Then f is continuous on D.*

Let $D \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on D. As with numerical series, we define the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$ by $s_m = \sum_{n=1}^{m} f_n$, and we say that the series $\sum_{n=1}^{\infty} f_n$ converges if the sequence of partial sums $\{s_m\}_{m=1}^{\infty}$ converges.

THEOREM 4.27. *Let $D \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on D. Suppose that there are numbers $c_n \in \mathbb{R}$ such that $|f_n| \leqslant c_n$, and such that $\sum_{n=1}^{\infty} c_n$ converges. Then the series $\sum_{n=1}^{\infty} f_n$ converges to a continuous function.*

CHAPTER 5

Integration and Differentiation

1. The Lower and Upper Intergals

We will use the notation $\{x_j\}_{j=0}^n$ for a *finite sequence*, i.e., a function $x\colon N_0^n \to \mathbb{R}$, where n is some positive integer, and $N_0^n = \{j \in \mathbb{Z}\colon\ 0 \leqslant j \leqslant n\}$. As before, a finite sequence $\{x_j\}_{j=0}^n$ is *increasing* if $x_{j-1} < x_j$ for all $1 \leqslant j \leqslant n$.

Definition 5.1. Let $n \in \mathbb{N}$. An increasing finite sequence $P = \{x_j\}_{j=0}^n$ with $x_0 = a$, and $x_n = b$ is called a *partition* of $[a, b]$. We denote the set of all partitions of $[a, b]$ by $\mathcal{P}[a, b]$. If $P = \{x_j\}_{j=1}^n \in \mathcal{P}[a, b]$, we define $\Delta_j(P) = x_j - x_{j-1}$. If $P_1, P_2 \in \mathcal{P}[a, b]$, we say that P_1 is a *refinement* of P_2, written $P_1 \supset P_2$, if the image of P_2 as a function is contained in the image of P_1.

Theorem 5.1. *Let $P_1, P_2 \in \mathcal{P}[a, b]$. Then there is $P \in \mathcal{P}[a, b]$ such that $P \supset P_1$, and $P \supset P_2$.*

A partition P satisfying $P \supset P_1$, and $P \supset P_2$, as given by this theorem, is called a *common refinement* of P_1 and P_2.

Definition 5.2. Let f be a bounded function defined on $[a, b]$, and let $P \in \mathcal{P}[a, b]$. We define for each $1 \leqslant j \leqslant n$:

$$L(f, P) = \sum_{j=1}^{n} m_j(f, P)\Delta_j(P), \qquad U(f, P) = \sum_{j=1}^{n} M_j(f, P)\Delta_j(P),$$

where

$$m_j(f, P) = \inf_{[x_{j-1}, x_j]} f, \qquad M_j(f, P) = \sup_{[x_{j-1}, x_j]} f.$$

We define the *lower integral* and the *upper integral* of f over $[a, b]$ by:

$$\underline{\int_a^b} f(x)\, dx = \sup\bigl\{L(f, P)\colon\ P \in \mathcal{P}[a, b]\bigr\},$$

$$\overline{\int_a^b} f(x)\, dx = \inf\bigl\{U(f, P)\colon\ P \in \mathcal{P}[a, b]\bigr\}.$$

Theorem 5.2. *Let f be a bounded function on $[a, b]$, and let $P \in \mathcal{P}[a, b]$, then $L(f, P) = -U(-f, P)$.*

Theorem 5.3. *Let f be a bounded function on $[a, b]$, and let $P_1, P_2 \in \mathcal{P}[a, b]$ satisfy $P_1 \supset P_2$. Then, $L(f, P_2) \leqslant L(f, P_1)$, and $U(f, P_1) \leqslant U(f, P_2)$.*

Theorem 5.4. *Let f be a bounded function on $[a, b]$, and let $P_1, P_2 \in \mathcal{P}[a, b]$, then $L(f, P_1) \leqslant U(f, P_2)$.*

THEOREM 5.5. *Let f be a bounded function on $[a,b]$, then*

$$\underline{\int_a^b} f\,dx \leqslant \overline{\int_a^b} f\,dx.$$

2. The Riemann Integral and its Properties

DEFINITION 5.3. Let f be a bounded function on $[a,b]$. We say that f is *Riemann integrable*, written $f \in \mathcal{R}[a,b]$, if the lower and upper integrals of f over $[a,b]$ coincide. In this case, we denote their common value by:

$$\int_a^b f(x)\,dx.$$

THEOREM 5.6. *Let $f \in \mathcal{R}[a,b]$. Then, for every $\varepsilon > 0$ there is $P \in \mathcal{P}[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

THEOREM 5.7. *Let f be a bounded function on $[a,b]$. Suppose that for every $\varepsilon > 0$ there is $P \in \mathcal{P}[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

Then $f \in \mathcal{R}[a,b]$.

THEOREM 5.8. *Suppose that f is continuous on $[a,b]$. Then $f \in \mathcal{R}[a,b]$.*

THEOREM 5.9. *Suppose that $f \in \mathcal{R}[a,b]$, and let $c \in \mathbb{R}$. Then $cf \in \mathcal{R}[a,b]$ and*

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

DEFINITION 5.4. A function $f\colon D \to \mathbb{R}$ is *non-decreasing* on D if for every $x, y \in D$ such that $x \leqslant y$, there holds $f(x) \leqslant f(y)$. We say that f is *non-increasing* on D is $-f$ is non-decreasing. A function $f\colon D \to \mathbb{R}$ is *monotonic* if f is either non-decreasing or non-increasing on $[a,b]$.

THEOREM 5.10. *Suppose that f is monotonic on $[a,b]$. Then $f \in \mathcal{R}[a,b]$.*

THEOREM 5.11. *Suppose that $f, g \in \mathcal{R}[a,b]$. Then $f+g \in \mathcal{R}[a,b]$ and*

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_c^b g(x)\,dx.$$

THEOREM 5.12. *Let $f \in \mathcal{R}[a,b]$ and $a < c < b$. Then, we have $f \in \mathcal{R}[a,c] \cap \mathcal{R}[c,b]$, and:*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

THEOREM 5.13. *Suppose that $f \in \mathcal{R}[a,b]$, and $f \geqslant 0$. Then*

$$\int_a^b f(x)\,dx \geqslant 0.$$

THEOREM 5.14. *Suppose that $f, g \in \mathcal{R}[a,b]$, and $f \leqslant g$. Then*

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

THEOREM 5.15. *Suppose that $f \in \mathcal{R}[a,b]$. Then $|f| \in \mathcal{R}[a,b]$, and*

$$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx.$$

THEOREM 5.16. *If $f \in \mathcal{R}[a,b]$, and $|f| \leqslant M$, then*

$$\left| \int_a^b f(x)\,dx \right| \leqslant M(b-a).$$

THEOREM 5.17. *Let $f \in \mathcal{R}[a,b]$. Define $F\colon [a,b] \to \mathbb{R}$ by:*

$$F(x) = \int_a^x f(t)\,dt,$$

for each $x \in [a,b]$. Then F is continuous on $[a,b]$.

THEOREM 5.18. *Let $f_n \in \mathcal{R}[a,b]$, and suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Then $f \in \mathcal{R}[a,b]$, and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

3. The Derivative

DEFINITION 5.5. Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, and $a \in \operatorname{int} D$. Define the function $f_a\colon D \setminus \{a\} \to \mathbb{R}$ by:

$$f_a(x) = \frac{f(x)-f(a)}{x-a}. \tag{5.1}$$

If the function f_a has the limit m at a, we say that m is *the derivative of f at a*, and write:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}. \tag{5.2}$$

If f has a derivative at a, we say that f is *differentiable at a*. If D is open, and f is differentiable at each $a \in D$, then we say that f is differentiable on D.

THEOREM 5.19. *Let $f\colon D \to \mathbb{R}$ be differentiable at $a \in \operatorname{int} D$. Then f is continuous at a.*

THEOREM 5.20. *Let $f\colon D \to \mathbb{R}$ be differentiable at $a \in \operatorname{int} D$, and let $c \in \mathbb{R}$. Then the function $cf\colon D \to \mathbb{R}$ is differentiable at a, and*

$$(cf)'(a) = cf'(a). \tag{5.3}$$

THEOREM 5.21. *Let $f, g\colon D \to \mathbb{R}$ be differentiable at $a \in \operatorname{int} D$. Then the function $f+g$ is differentiable at a, and*

$$(f+g)'(a) = f'(a) + g'(a). \tag{5.4}$$

THEOREM 5.22. *Let $f, g\colon D \to \mathbb{R}$ be differentiable at $a \in \operatorname{int} D$. Then the function fg is differentiable at a, and*

$$(fg)'(a) = f'(a)\,g(a) + f(a)\,g'(a). \tag{5.5}$$

THEOREM 5.23. *For each* $1 \leqslant k \leqslant n$, *let* $c_k \in R$, *and let* $p\colon \mathbb{R} \to \mathbb{R}$ *be the polynomial given by:*

$$p(x) = \sum_{k=0}^{n} c_k\, x^k. \tag{5.6}$$

Let $a \in \mathbb{R}$, *then* p *is differentiable at* a, *and*

$$p'(a) = \sum_{k=1}^{n} k\, c_k a^{k-1}. \tag{5.7}$$

4. The Mean Value Theorem

THEOREM 5.24. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and differentiable on* (a,b). *Suppose that there is* $c \in (a,b)$, *such that* $f(c) = \sup_{(a,b)} f$. *Then* $f'(c) = 0$.

THEOREM 5.25. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and differentiable on* (a,b). *Suppose also that* $f(a) = f(b) = 0$. *Then there is* $c \in (a,b)$ *such that*

$$f'(c) = 0. \tag{5.8}$$

THEOREM 5.26. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and differentiable on* (a,b). *Then there is* $c \in (a,b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{5.9}$$

THEOREM 5.27. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and differentiable on* (a,b). *Suppose also that*

$$f'(c) = 0, \tag{5.10}$$

for all $c \in (a,b)$. *Then, there is* $d \in \mathbb{R}$, *such that* $f(x) = d$ *for all* $x \in [a,b]$.

THEOREM 5.28. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and differentiable on* (a,b). *Suppose also that*

$$f'(c) \geqslant 0, \tag{5.11}$$

for each $c \in (a,b)$. *Then* f *is non-decreasing on* $[a,b]$.

5. Integration and Differentiation

THEOREM 5.29. *Let* $f \in \mathcal{R}[a,b]$. *Define* $F\colon [a,b] \to \mathbb{R}$ *by:*

$$F(x) = \int_a^x f(t)\, dt,$$

for $x \in [a,b]$. *If* f *is continuous at* $c \in [a,b]$, *then* F *is differentiable at* c, *and*

$$F'(c) = f(c).$$

THEOREM 5.30. *Let* $f \in \mathcal{R}[a,b]$, *and suppose there is a continuous function* $F\colon [a,b] \to \mathbb{R}$ *such that* $F'(x) = f(x)$ *for each* $x \in (a,b)$. *Then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$